A NEW DIMENSION FUNCTION

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ABSTRACT. A new inductive dimension function, Hind, is defined for hereditarily normal spaces. The countable and locally finite sum theorems for Hind are proved for hereditarily normal spaces. It is shown that Hind = Ind on the class of totally normal spaces.

1. Introduction. The finite sum theorem for the strong inductive dimension Ind does not hold in general for compact Hausdorff spaces. The counterexample is due to Lokucievskii [4], [5, Example 16-6]. It is an open question whether the finite sum theorem holds for hereditarily normal spaces. The positive results which are due to Dowker are as follows. In the class of totally normal spaces the countable sum theorem, the open subset theorem and the subset theorem for Ind hold [1], [5, §11]. There is also a locally finite sum theorem for Ind in totally normal spaces [3], [5, Lemma 25-3]. The proofs of these results are strongly interrelated.

In this note we introduce a new inductive dimension function, Hind, for hereditarily normal spaces for which the sum theorems can be proved without relying on an open subset theorem. These theorems are valid in the class of hereditarily normal spaces and the proofs are straightforward. It is to be noted that the (open) subset theorem for Ind as well as Hind does not hold in the class of hereditarily normal spaces. The counterexample which is due to Filippov [2] will be discussed in §2.

The dimension function Hind agrees with Ind on the class of totally normal spaces. Also, Hind X = Ind X for every hereditarily normal space X with Hind X ≤ 1. So by the defining of Hind we have singled out the property which is crucial for the sum theorems of the inductive dimension.

It is an open question whether Hind and Ind agree on the class of hereditarily normal spaces. This question is related to that mentioned above. A negative answer to the former question provides a negative answer to the latter.

2. Definition of Hind. The huge inductive dimension Hind is defined for every hereditarily normal space X as follows:

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(1) Hind $X = -1$ if and only if $X = \emptyset$;

(2) For each integer $n \geq 0$, Hind $X \leq n$ provided that for each pair of closed subsets $F$ and $G$ with Hind$(F \cap G) \leq n - 1$ there is a pair of closed subsets $K$ and $L$ such that $F \setminus G \subseteq K \setminus L$, $G \setminus F \subseteq L \setminus K$, $K \cup L = X$ and Hind$(K \cap L) \leq n - 1$.

Hind $X = n$, $n = 0, 1, 2, \ldots, \infty$, are defined as usual.

Observe that in (2) the set $K \cap L$ separates $F \setminus G$ and $G \setminus F$ in $X$. This proves the "only if"-part of the following proposition.

**Proposition 1.** For each integer $n \geq 0$, Hind $X \leq n$ if and only if for each pair of closed subsets $F$ and $G$ with Hind$(F \cap G) \leq n - 1$ there is a closed subset $S$ such that $F \setminus G$ and $G \setminus F$ are separated by $S$ and Hind $S \leq n - 1$.

**Proof.** The "if"-part is proved as follows. Let $F$ and $G$ be closed subsets with Hind$(F \cap G) \leq n - 1$ and let $S$ be a closed subset which separates $F \setminus G$ and $G \setminus F$ and which has Hind $\leq n - 1$. Then $X \setminus S = U \cup V$ where $U$ and $V$ are disjoint open sets containing $F \setminus G$ and $G \setminus F$ respectively. Then $K = U \cup S$ and $L = V \cup S$ satisfy the conditions required in the definition of Hind $X \leq n$.

**Corollary 1.** Let $n = -1$ or 0, Hind $X = n$ if and only if Ind $X = n$ for every hereditarily normal space $X$.

The easy inductive proofs of the following propositions are omitted.

**Proposition 2.** Ind $\leq$ Hind on the class of hereditarily normal spaces.

**Proposition 3.** Let $C$ be a closed subset of a hereditarily normal space $X$. Then Hind $C \leq$ Hind $X$.

From Corollary 1 and Proposition 2 it follows that Hind $X = \text{Ind} X$ for every hereditarily normal space $X$ with Hind $X \leq 1$.

**Example 1.** Filippov [2] has given an example of a hereditarily normal and zero-dimensional space $X$ which contains subspaces $X_n$, $n = 1, 2, \ldots$, with Ind $X_n = n$. The construction of the example is based on the assumption of the existence of a Suslin tree. In view of Corollary 1 and Proposition 2 the same example shows the failure of the subset theorem for Hind in the class of hereditarily normal spaces. It is to be observed that the open subset theorem and the subset theorem for Ind are equivalent in the class of hereditarily normal spaces [1], [5, §11-2]. So there is also no open subset theorem for Hind in the class of hereditarily normal spaces.
3. The sum theorems. The proofs of the sum theorems are based on the following lemmas.

**Lemma 1.** Let $F$ and $G$ be closed subsets of a hereditarily normal space $X$. Then in $X$ there exist closed neighborhoods $U$ and $V$ of $F \setminus G$ and $G \setminus F$, respectively, such that $U \cap V = F \cap G$.

**Proof.** $F \setminus G$ and $G \setminus F$ are disjoint closed subsets of the normal subspace $X_1 = X \setminus (F \cap G)$. So in $X_1$ there are disjoint closed neighborhoods $U_1$ and $V_1$ of $F \setminus G$ and $G \setminus F$ respectively. As $X_1$ is open, $U_1$ and $V_1$ are also neighborhoods in $X$. Now let $U = F \cup U_1$ and $V = G \cup V_1$. As $\text{cl}_X U_1 \subset U_1 \cup (F \cap G)$, $U$ is closed in $X$. Similarly $V$ is closed.

$U \cap V = F \cap G$, because $U \cap V \subset F \cap G$ and also $V \cap U \subset F \cap G$.

**Lemma 2.** Let $F$ and $G$ be closed subsets of a hereditarily normal space $X$ and let $Y$ be a subset of $X$ such that $Z = F \cup G \cup Y$ is closed in $X$. Suppose $\text{Ind} \ Y \leq n$ and $\text{Ind}(F \cap G \cap Y) \leq n - 1$. Then there exist closed subsets $K$ and $L$ of $X$ and a closed subset $S$ of $Y$ such that

1. $F \setminus G \subset K \setminus L$ and $G \setminus F \subset L \setminus K$;
2. $K \cup L = Z$;
3. $K \cap L = (F \cap G) \cup S$;
4. $\text{Ind} S \leq n - 1$.

**Proof.** By Lemma 1 in $X$ there are closed neighborhoods $U$ and $V$ of $F \setminus G$ and $G \setminus F$, respectively, such that $U \cap V = F \cap G$. By virtue of Proposition 1 there is a partition $\{Q, R, S\}$ of $Y$, where $Q$ and $R$ are open and $S$ is closed in $Y$, such that $(U \setminus V) \cap Y \subset Q$, $(V \setminus U) \cap Y \subset R$ and $\text{Ind} S \leq n - 1$. Now let $K = F \cup Q \cup S$ and $L = G \cup R \cup S$. First observe that $V \setminus U$ is a neighborhood of $G \setminus F$. As $(Q \cup S) \cap (V \setminus U) = \emptyset$, $\text{cl}_X (Q \cup S) \cap (G \setminus F) = \emptyset$. Hence $\text{cl}_X (Q \cup S) \subset K$, because $Z$ is closed. It follows that $K$ is closed in $X$. Similarly, $L$ is closed. We also have $(Q \cup S) \cap G \subset F \cap G$ and, similarly, $(R \cup S) \cap F \subset G \cap F$. So $K \cap L = (F \cap G) \cup S$. Clearly $S \cap (F \cup G) \subset F \cap G$. It follows that $K \setminus L \supset F \setminus (K \cap L) = F \setminus G$. Now it is easily seen that $(1)$–$(4)$ are satisfied.

**Remark 1.** If in Lemma 2 we suppose that $\text{Ind} Y \leq n$ and $F \cap G \cap Y = \emptyset$ (instead of $\text{Ind} Y \leq n$ and $\text{Ind}(F \cap G \cap Y) \leq n - 1$), in the same way it can be shown that there exist closed subsets $K$ and $L$ of $X$ and a closed subset $S$ of $Y$ satisfying $(1)$, $(2)$, $(3)$ and $\text{Ind} S \leq n - 1$.

We now prove the countable sum theorem.

**Theorem 1.** Let $\{X_i\}_{i = 1, 2, \cdots}$ be a countable closed cover of the
hereditarily normal space $X$, such that $\Hind X_i \leq n$, $i = 1, 2, \ldots$. Then $\Hind X \leq n$.

**Proof.** The proof is by induction on $n$. The theorem is obvious for $n = -1$. Suppose it has been proved for $n - 1$. Let $K_0$ and $L_0$ be closed subsets of $X$ such that for $D_0 = K_0 \cap L_0$ we have $\Hind D_0 \leq n - 1$. Inductively on $i$ we shall define closed subsets $D_i$, $K_i$ and $L_i$ of $X$, $i = 1, 2, \ldots$, such that for each $i$:

1. $K_{i-1} \setminus L_{i-1} \subseteq \text{int}_X K_i \setminus L_i$ and $L_{i-1} \setminus K_{i-1} \subseteq \text{int}_X L_i \setminus K_i$;
2. $X_i \subseteq K_i \cup L_i$;
3. $D_i = K_i \cap L_i$;
4. $D_{i-1} \subseteq D_i$; and
5. $\Hind D_i \leq n - 1$.

Suppose $D_j$, $K_j$ and $L_j$ have already been defined for $j = 0, 1, \ldots, i - 1$ satisfying (1)–(5). In view of (3) and Lemma 1 there are closed neighborhoods $U$ and $V$ of $K_{i-1} \setminus L_{i-1}$ and $L_{i-1} \setminus K_{i-1}$, respectively, such that $U \cap V = D_{i-1}$.

By applying Lemma 2 (with $F$, $G$ and $Y$ replaced by $U$, $V$ and $X_i$, respectively) we get closed sets $K_i$, $L_i$ and $S$ such that, if we define $D_i = K_i \cap L_i$, then (1)–(3) above are satisfied, $D_i = D_{i-1} \cup S$ and $\Hind S \leq n - 1$. Clearly (4) is satisfied and (5) follows from the induction hypothesis.

Finally let $U = \bigcup \{K_i \setminus L_i \mid i = 1, 2, \ldots\}$, $V = \bigcup \{L_i \setminus K_i \mid i = 1, 2, \ldots\}$ and $D = \bigcup \{D_i \mid i = 1, 2, \ldots\}$. Then $\{U, V, D\}$ is a partition of $X$ by virtue of (1)–(4). $U$ and $V$ are open in view of (1). Hence $D$ is closed. By the induction hypothesis and (5) we get $\Hind D \leq n - 1$. By Proposition 1 we have $\Hind X \leq n$, since $K_0 \setminus L_0 \subseteq U$ and $L_0 \setminus K_0 \subseteq V$ in view of (1).

The locally finite sum theorem is as follows.

**Theorem 2.** Let $\{X_\alpha \mid \alpha \in A\}$ be a locally finite closed cover of a hereditarily normal space $X$, such that $\Hind X_\alpha \leq n$ for each $\alpha \in A$. Then $\Hind X \leq n$.

**Proof.** The proof is by induction on $n$. The theorem is obvious for $n = -1$. Suppose it has been proved for $n - 1$. Let $K_0$ and $L_0$ be subsets of $X$ such that for $D_0 = K_0 \cap L_0$ we have $\Hind D_0 \leq n - 1$. Suppose $\{X_\alpha \mid \alpha \in A\}$ is well ordered as $\{X_\alpha \mid 1 \leq \alpha < \tau\}$. Let $Z_\alpha = \bigcup \{X_\beta \mid 1 \leq \beta < \alpha\}$ for $1 \leq \alpha < \tau$. Let $Z_0 = \emptyset$. By transfinite induction on $\alpha$ we shall define closed subsets $K_\alpha$, $L_\alpha$ and $D_\alpha$ of $X$ such that for all $\alpha$, $\beta$, $\gamma < \tau$:

1. $K_\beta \setminus L_\beta \subseteq K_\gamma \setminus L_\gamma$ and $L_\beta \setminus K_\beta \subseteq L_\gamma \setminus K_\gamma$ if $\beta < \gamma$;
A NEW DIMENSION FUNCTION

(2) $K_\alpha \cup L_\alpha = Z_\alpha \cup (K_0 \cup L_0)$;
(3) $D_\alpha = K_\alpha \cap L_\alpha$;
(4) $D_\beta \subseteq D_\gamma$ for $\beta < \gamma$; and
(5) $\text{Hind } D_\alpha \leq n - 1$.

Suppose $D_\beta$, $K_\beta$ and $L_\beta$ have been defined for all $\beta < \alpha$ satisfying (1)–(5) for ordinals less than $\alpha$.

Let $F = \bigcup \{K_\beta : \beta < \alpha\}$, $G = \bigcup \{L_\beta : \beta < \alpha\}$ and $T = \bigcup \{D_\beta : \beta < \alpha\}$. In view of (1) and (3) we have $F \cap G = T$. Observe that $T = (K_0 \cap L_0) \cup \{D_\beta \cap X_\beta : 1 \leq \beta < \alpha\}$, because, if $x \in T \setminus (K_0 \cap L_0)$, then for the minimal $\gamma < \alpha$ with $x \in X_\gamma$ we also have $x \in D_\gamma$ in view of (1) and (3). Note that $\{D_\beta \cap X_\beta : \beta < \alpha\}$ is a locally finite closed collection. In view of Proposition 3 and the induction hypothesis, $\text{Hind } T \leq n - 1$. It also follows that $T$ is closed. Similarly $F$ and $G$ can be shown to be closed.

By applying Lemma 2 (with $Y$ replaced by $X_\alpha$) we get closed sets $K_\alpha$, $L_\alpha$ and $S$ such that, if we define $D_\alpha = K_\alpha \cap L_\alpha$, then (1)–(3) above are satisfied for ordinals less than or equal to $\alpha$, $D_\alpha = T \cup S$ and $\text{Hind } S \leq n - 1$. Trivially (4) holds for ordinals not exceeding $\alpha$ and (5) follows from Theorem 1.

Finally let $F^* = \bigcup \{K_\alpha : \alpha < \nu\}$, $G^* = \bigcup \{L_\alpha : \alpha < \nu\}$ and $T^* = \bigcup \{D_\alpha : \alpha < \nu\}$. In the same way as has been done for $F$, $G$ and $T$, it is shown that $F^*$, $G^*$ and $T^*$ are closed, that $T^* = F^* \cap G^*$ and that $\text{Hind } T^* \leq n - 1$. The construction ensures that $K_0 \setminus L_0 \subseteq F^* \setminus G^*$ and $L_0 \setminus K_0 \subseteq G^* \setminus F^*$ in view of (1). Hence $\text{Hind } X \leq n$.

The following corollary will be useful in establishing the equality of Ind and Hind in the class of totally normal spaces. This result is also in [1].

**Corollary 2.** Let the hereditarily normal space $X$ be the disjoint union of a closed subspace $X_1$ and an open subspace $X_2$. If $\text{Ind } X_1 \leq n$ and $\text{Ind } X_2 \leq n$, then $\text{Ind } X \leq n$.

**Proof.** The proof is by induction and the case $n = -1$ is obvious. Assume the theorem has been proved for $n - 1$. Let $K_0$ and $L_0$ be disjoint closed subsets of $X$. Let $D_1$, $K_1$ and $L_1$ be constructed as in the proof of Theorem 2 in such a way that $\text{Ind } D_1 \leq n - 1$. It is to be noted that $D_1 \cap X_2 = \emptyset$. In view of Remark 1 by Lemma 2 (with $F$, $G$ and $Y$ replaced by $K_1$, $L_1$ and $X_2$ respectively) we get closed subsets $K$ and $L$ of $X$ and a closed subset $S$ of $X_2$ such that $K_0 \subseteq K \setminus L$, $L_0 \subseteq L \setminus K$, $K \cup L = X$, $K \cap L = D_1 \cup S$ and $\text{Ind } S \leq n - 1$. By the induction hypothesis $\text{Ind } (K \cap L) \leq n - 1$. It follows that $\text{Ind } X \leq n$.

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4. Results for totally normal spaces. Recall that a normal space $X$ is totally normal if each open subset $U$ of $X$ admits a cover $\mathcal{U}$ such that each $V \in \mathcal{U}$ is a cozero-subset in $X$ and $\mathcal{U}$ is locally finite in $U$. As a cozero-subset of $X$ is an $F_\sigma$-subset of $X$, by Proposition 3 and Theorems 1 and 2 we have that $\text{Hind } U \leq \text{Hind } X$ whenever $U$ is an open subset of a totally normal space $X$.

Thus the following proposition has been proved.

**Proposition 4.** Let $X$ be a totally normal space and let $U$ be an open subset of $X$. Then $\text{Hind } U \leq \text{Hind } X$.

We shall now show that $\text{Ind}$ and $\text{Hind}$ coincide on the class of totally normal spaces.

**Theorem 3.** $\text{Ind} = \text{Hind}$ on the class of totally normal spaces.

**Proof.** In view of Proposition 2 we need only prove that $\text{Hind} \leq \text{Ind}$ on the class of totally normal spaces. By induction on $n$ we shall prove the following statement.

$E(n)$: If $\text{Ind } X \leq n$, then $\text{Hind } X \leq n$ for every totally normal space $X$.

We assume $E(n)$ has been proved for $-1, 0, \ldots, n-1$ (cf. Corollary 1). Let $F$ and $G$ be closed subsets of $X$ such that $\text{Hind } (F \cap G) \leq n - 1$. Define $Y = X \setminus (F \cap G)$. In view of the open subset theorem for $\text{Ind}$ on the class of totally normal spaces [5, §11-5] (see also below) $\text{Ind } Y \leq n$. By Lemma 2 and Remark 1 there exist closed subsets $K$ and $L$ of $X$ and a closed subset $S$ of $Y$ such that

1. $F \setminus G \subset K \setminus L$ and $G \setminus F \subset L \setminus K$;
2. $K \cup L = X$;
3. $K \cap L = (F \cap G) \cup S$;
4. $\text{Ind } S \leq n - 1$.

By Corollary 2 and Proposition 3 we have $\text{Ind } (K \cap L) \leq n - 1$. By virtue of the induction hypothesis $\text{Hind } (K \cap L) \leq n - 1$, whence $\text{Hind } X \leq n$.

**Remark on the open subset theorem for $\text{Ind.}$** A proof of the open subset theorem for $\text{Ind}$, which is used in the proof of Theorem 3, can be based on the results in §3 and the induction hypothesis. For completeness sake we shall indicate such a proof. Combined with the results of §3, this will give a new proof of the sum theorems for $\text{Ind}$.

Assuming $E(n - 1)$ we first prove

$A(n)$: Let $\{X_\alpha \mid \alpha \in A\}$ be a locally finite closed collection of totally normal space $X$ such that $\{\text{int } X_\beta \mid \alpha \in A\}$ is a cover of $X$ and $\text{Ind } X_\alpha \leq n$ for each $\alpha \in A$. Then $\text{Ind } X \leq n$.

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Proof of $A(n)$. (int, cl and $B$ denote the interior, closure and boundary operator in $X$.) Let $\{F_\alpha | \alpha \in A\}$ be a closed cover of $X$ such that $F_\alpha \subset \text{int } X_\alpha$. Suppose $F$ is closed in $X$ and $V$ is an open neighborhood of $F$. Choose open subsets $W_\alpha$ such that $\text{Ind } B(W_\alpha) \leq n - 1$ and $F_\alpha \cap F \subset W_\alpha \subset \text{cl } W_\alpha \subset V \cap \text{int } X_\alpha$. By $E(n-1)$ and Theorem 2 we get $\text{Hind } \bigcup \{B(W_\alpha) | \alpha \in A\} \leq n - 1$. Let $W = \bigcup \{W_\alpha | \alpha \in A\}$. Then $F \subset W \subset V$ and $\text{Hind } B(W) \leq n - 1$, because $B(W) \subset \bigcup \{B(W_\alpha) | \alpha \in A\}$ in view of the local finiteness of the last collection. By Proposition 3 and $E(n-1)$ it follows that $\text{Ind } B(W) \leq n - 1$, whence $\text{Ind } X \leq n$.

In a standard fashion from $A(n)$ one can deduce

$O(n)$: If $U$ is an open subset of a totally normal space $X$ with $\text{Ind } X \leq n$, then $\text{Ind } U \leq n$.

REFERENCES


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