

CANTOR SETS AND HOMOTOPY CONNECTEDNESS OF MANIFOLDS¹

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ABSTRACT. We prove that a topological manifold M of dimension n is $(n - 2)$ -connected if each Cantor set in M is contained in an open n -ball of M . An immediate consequence is that a compact manifold N of dimension n ($n \geq 5$) is homeomorphic to the n -sphere if and only if every Cantor set of N is contained in an open n -ball of N . This consequence generalizes a 3-dimensional theorem of Doyle and Hocking.

1. Introduction. The research in this note was undertaken in an attempt to prove the following conjecture suggested to us by J. W. Cannon.

Conjecture 1.1. A compact manifold M of dimension n ($n \geq 3$) is an n -sphere if and only if every Cantor set of M lies in an open n -ball of M .

The conjecture for the case $n = 3$ was shown to be true by Doyle and Hocking [6, Theorem 6]. We give an affirmative answer for $n \geq 5$ by proving that if every Cantor set in a manifold M lies in an open n -ball, then M is $(n - 2)$ -connected. The conjecture for $n \geq 5$ then follows from the topological Poincaré theorem. Our results imply that a compact 4-manifold in which every Cantor set lies in an open 4-ball is of the same homotopy type as a 4-sphere.

R. P. Osborne has treated similar questions. His ideas reported in [9] were helpful to us. We express our indebtedness for the friendship, encouragement, and instruction of J. W. Cannon.

2. Manifolds in which Cantor sets lie in balls. The only fact we will need to know about Cantor sets is the following theorem.

Theorem 2.1. *Let N be a regular neighborhood of an unknotted simple*

Presented to the Society, January 18, 1974; received by the editors May 24, 1974.
AMS (MOS) subject classifications (1970). Primary 54F65, 57A10, 57A15;

Secondary 57A30, 57A35.

Key words and phrases. Cantor set, homotopy connected, engulfing, characterization of n -sphere.

¹This work constitutes part of the author's doctoral thesis under Professor J. W. Cannon at the University of Wisconsin, Madison.

²Research supported in part by NDEA Title IV Fellowship.

closed curve J in Euclidean n -space E^n ($n \geq 3$). Then there exists a Cantor set C in the complement of N such that the inclusion from N into $E^n - C$ induces a monomorphism on fundamental group.

R. P. Osborne has shown the existence of Cantor sets [9, Theorem 2.5] which imply that the above theorem is correct. We have given an independent proof [15, Theorem 5.3] which implies that the Cantor set of Theorem 2.1 could be chosen to be a Blankinship Cantor set [3]. Our proof is somewhat similar to a proof [7] given by Eaton in which he showed the Blankinship Cantor sets to be wild. However, Eaton's proof does not show that the fundamental group of the complement of the Blankinship Cantor set in E^n is nontrivial.

Theorem 2.2. *Suppose M is a manifold of dimension n ($n \geq 3$) such that every Cantor set of M is contained in an open n -ball of M . Then M is simply connected.*

Proof. Let K be a loop in M which is the union of arcs $\sigma_1, \dots, \sigma_m$ such that each σ_i is linearly embedded in some open n -ball U_i of M . There exist PL 2-spheres S_i^2 in each U_i such that $\sigma_i \subset S_i^2$ and S_i^2 is the join of $\text{Bd } \sigma_i$ and a simple closed curve Σ_i . Let B be a collared n -ball of M which contains the end points of the σ_i in its interior. Let G_i be a regular neighborhood of the end points of σ_i in S_i^2 such that $G_i \subset B$. We let N_i be a regular neighborhood of Σ_i in U_i such that $S_i^2 - G_i \subset N_i$. By Theorem 2.1 there exists a Cantor set C_i in $U_i - N_i$ such that the inclusion from N_i into $U_i - C_i$ induces a monomorphism on fundamental group.

If we identify the ball B to a point we get M/B which is homeomorphic to M [4]. Let P be the identification map from M to M/B . The compact 0-dimensional set $P(\bigcup C_i \cup B)$ is contained in a Cantor set C in M/B . By the hypothesis of the theorem the Cantor set C is contained in an open n -ball U' of M/B . The set $U = P^{-1}(U')$ is an open n -ball [4] in M which contains $\bigcup C_i \cup B$.

We let W be a closed collared ball in U such that $\bigcup C_i \cup B \subset W$. Triangulate each S_i^2 so finely that if a simplex of S_i^2 intersects W then it is contained in U . Let P_i be the union of all the closed simplexes of S_i^2 which miss W . The polyhedron P_i is contained in $S_i^2 - G_i$ and hence in N_i . If P_i separates the end points of σ_i in S_i^2 , then P_i contains a simple closed curve γ_i which is homotopic to Σ_i in N_i . Since γ_i is in the complement of

W , γ_i is homotopically trivial in $M - W$. Since U_i is simply connected at infinity, γ_i is homotopically trivial in $U_i - C_i$ which is a contradiction to the choice of C_i . Thus we may conclude that P_i does not separate the end points of σ_i in S_i^2 .

Let σ'_i be an arc connecting the end points of σ_i in $S_i^2 - P_i$. Let K' be the loop formed by taking the union of the σ'_i in the appropriate order so that K is homotopic to K' . The loop K' is contained in U and hence is homotopically trivial. Thus K is homotopically trivial in M , and M is simply connected.

Doyle and Hocking [6, Theorem 7] claimed that Theorem 2.2 could be proved in the PL case ($n > 3$) by appropriate modifications of their 3-dimensional theorem. We were unable to find modifications of their techniques sufficient to prove Theorem 2.2; indeed, a number of rather surprising difficulties arise for $n > 3$ which are not at all apparent in the 3-dimensional case. We will now outline an alternative proof of the Doyle and Hocking 3-dimensional theorem.

Theorem 2.3 (Doyle and Hocking). *Suppose M is a compact 3-manifold such that every Cantor set of M lies in an open 3-ball of M . Then M is homeomorphic to the 3-sphere.*

Proof. Doyle and Hocking proved this theorem by showing that every polygonal simple closed curve of M is contained in a 3-ball of M and then quoting a theorem [2, Theorem 1] of Bing. We proceed alternatively and show that every finite polyhedral graph in M is in an open 3-ball. Then, using complementary 1-skeleta and Stallings' stretching technique [13], we find that M is the union of two open 3-balls and hence [4] is homeomorphic to the 3-sphere.

We assume M is triangulated [1], [8]. Let L be a finite polyhedral graph in M . With each 1-simplex σ_i of L we associate a 2-sphere S_i^2 as shown in Figure 1. By the proof of Theorem 2.2 we may assume that $\bigcup \delta_i \cup (L - \bigcup S_i^2)$ is contained in an open 3-ball U where, for each i , δ_i is an arc in S_i^2 joining the end points of the arc $S_i^2 \cap \sigma_i$. We take an isotopy of the 2-spheres S_i^2 which takes each δ_i onto $S_i^2 \cap \sigma_i$ and keeps the end points of δ_i fixed. By using collars on the S_i^2 we extend this to an isotopy of M which takes K into U . In accord with the remarks of the preceding paragraph, this completes the proof of Theorem 2.3.

Before we extend Theorem 2.2 we will need the following lemma.

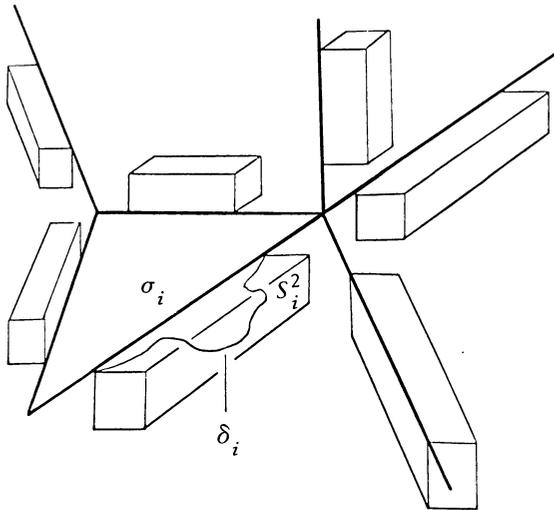


Figure 1

Lemma 2.1. Consider the $(n + 2)$ -sphere S^{n+2} ($n \geq 0$) as the join of S^1 and S^n . Suppose P is a polyhedron in $S^{n+2} - S^n$ such that the homomorphism $i_*: H_1(P) \rightarrow H_1(S^{n+2} - S^n)$ induced by the inclusion is trivial. Then the homomorphism $j_*: \tilde{H}_n(S^n) \rightarrow \tilde{H}_n(S^{n+2} - P)$ induced by the inclusion is trivial.

Proof. We assume familiarity with linking [11, pp. 257–268]. We will think of $S^n \subset S^{n+2}$ as both a point set and an n -cycle. Suppose S^n does not bound homologically in the complement of P . Then we can find a finite polyhedral graph Γ contained in P such that S^n does not bound homologically in the complement of Γ . This fact follows if we remove top dimensional simplexes of P one at a time and apply the Alexander addition theorem [14, Theorem 5.18, p. 60]. Furthermore we may assume that S^n bounds in the complement of any proper subgraph of Γ .

The graph Γ cannot be a tree. Hence we can find a pair (J, σ) where J is a simple closed curve in Γ and σ is a 1-simplex of J . By hypothesis J and S^n are not linked. By geometrical considerations S^n bounds in the complement of J . The graph Γ was chosen minimal; hence, S^n bounds in the complement of $\Gamma - \sigma$. By the Alexander addition theorem S^n bounds in the complement of Γ . Thus we have arrived at a contradiction and our lemma is proved.

Lemma 2.1 could be generalized by replacing S^n by an n -cycle Z^n of

S^{n+2} . We could then conclude that Z^n bounds in the complement of P .

Theorem 2.4. *Suppose M is a manifold of dimension n such that every Cantor set of M is contained in an open n -ball of M . Then M is $(n - 2)$ -connected; i. e., $\Pi_i(M) = 1$ ($0 \leq i \leq n - 2$).*

The Poincaré theorem [5, Corollary 1] and Theorem 2.4 together imply our desired generalization of Theorem 2.3:

Corollary 2.1. *Suppose M is a compact n -manifold ($n \geq 5$) such that every Cantor set of M lies in an open n -ball of M . Then M is homeomorphic to the n -sphere.*

Theorem 2.3 and Corollary 2.1 give the affirmative answer to Conjecture 1.1 for $n \neq 4$.

Proof of PL case. We will prove Theorem 2.3 in the case where M is a PL manifold and then indicate the changes necessary for the general case.

By Theorem 2.2 and an induction we may assume that, for some integer r in the range $1 \leq r \leq n - 3$, M is r -connected. Let $f: S^{r+1} \rightarrow M$ be a simplicial map. By the Hurewitz isomorphism theorem [12, Theorem 5, p. 398] it will be sufficient to show that the homomorphism $f_*: H_{r+1}(S^{r+1}) \rightarrow H_{r+1}(M)$ induced by f is trivial. The polyhedron $K = f(S^{r+1})$ is of dimension at most $r + 1$. Let $\sigma_1, \dots, \sigma_m$ be the $r + 1$ simplexes of K . As in the proof of Theorem 2.2 there exist open balls U_i of M and PL spheres S_i^{r+2} of dimension $r + 2$ in U_i such that $\sigma_i \subset S_i^{r+2} \subset U_i$ and S_i^{r+2} is the join of $\text{Bd } \sigma_i$ and a simple closed curve Σ_i . Since M is r -connected and $n - r \geq 3$, we can employ engulfing [13] to show that the r -skeleton of K lies in the interior of a collared n -ball B of M . Let G_i be a regular neighborhood of $\text{Bd } \sigma_i$ in S_i^{r+2} such that $G_i \subset B$. We let N_i be a regular neighborhood of Σ_i in U_i such that $S_i^{r+2} - G_i \subset N_i$. By Theorem 2.1 there exists a Cantor set C_i in $U_i - N_i$ such that the inclusion from N_i into $U_i - C_i$ induces a monomorphism on fundamental group. Repeating the argument of Theorem 2.2 we find that there is an open ball U which contains $\bigcup C_i \cup B$.

Again we let W be a closed collared ball in U such that $\bigcup C_i \cup B \subset W$. Triangulate each S_i^{r+2} so finely that if a simplex of S_i^{r+2} intersects W it is contained in U . Let P_i be the union of all closed simplexes of S_i^{r+2} which miss W . If the homomorphism $i_*: H_1(P_i) \rightarrow H_1(S_i^{r+2} - \text{Bd } \sigma_i)$ induced by the inclusion is not trivial, then there exists a loop γ_i in P_i which is homotopic to a nonzero multiple of Σ_i in N_i . As in the proof of Theorem 2.2 this would imply that the inclusion from N_i into $U_i - C_i$ does not induce a monomorphism

on fundamental group which is a contradiction to the choice of C_i . Hence by Lemma 2.1 $\text{Bd } \sigma_i$ bounds an $(r + 1)$ -chain K_i in $S_i^{r+2} - P_i \subset U$. Thus if α is the generator of $H_{r+1}(S^{r+1})$, then $f_*(\alpha)$ is homologous in M to a cycle of the form $\sum n_i K_i$ which lies in U and hence is trivial. This implies that f_* is trivial and completes the proof.

To prove Theorem 2.4 in the topological case we will need the following topological engulfing lemma which is a special case of a theorem [10, Theorem 4.12.1, pp. 200–201] formulated by Rushing.

Lemma 2.2. *Let M be a topological manifold of dimension n which is r -connected ($r \leq n - 3$). Let U be an open n -ball in M , A be a compact subset of U , and P be a finite locally tame polyhedron in M of dimension r . Then there is a homeomorphism h of M onto M such that $A \cup P \subset h(U)$.*

Lemma 2.3. *Let $H_0: P \rightarrow M$ be a map of a finite polyhedron P with triangulation K into an n -dimensional manifold M which is r -connected ($r \leq n - 3$). Then H_0 is homotopic to a map H_1 by an arbitrarily small homotopy such that $H_1(|K^r|)$ is contained in an open n -ball of M where $|K^r|$ is the carrier of the r -skeleton of K .*

Proof. The proof is by induction on r . For $r = 0$ we may set $H_1 = H_0$. By a subdivision of K and induction we may assume that, for each simplex $\sigma \in K$, $H_0(\sigma)$ is contained in an open n -ball U_σ of M of small diameter and that $H_0(|K^{r-1}|)$ is contained in an open n -ball U of M .

Let $V_\sigma \subset U_\sigma$ be the open set $\bigcap \{U_\nu | \nu \in K \text{ and } H_0(\sigma) \subset U_\nu\}$. We construct a small homotopy H between H_0 and H_1 as follows. We set $H_t(|K^{r-1}|) = H_0(|K^{r-1}|)$. If σ is an r -simplex of K , then $H_1|\sigma: \sigma \rightarrow V_\sigma \subset U_\sigma$ is a simplicial approximation of H_0 relative to the boundary and $H(\sigma \times I) \subset V_\sigma$. Notice that there is a finite polyhedron $P_\sigma \subset U_\sigma$ of dimension r such that $H_1(\sigma)$ is contained in the union of a compact subset of U and P_σ .

We finish the definition of H by assuming inductively that H is defined on $(P \times \{0\}) \cup (|K^m| \times I)$ where $m \geq r$ and that for each $\sigma \in K^m$, $H(\sigma \times I) \subset V_\sigma$. Let ν be an $(m + 1)$ -simplex. The set $H((\text{Bd } \nu) \times I)$ is contained in V_ν ; hence, it is possible to extend H to $\nu \times I$ such that $H(\nu \times I) \subset V_\nu$.

Since $H_1(|K^{r-1}|) \subset U$, Lemma 2.2 may be used inductively to obtain a homeomorphism h of M onto itself such that $H_1(|K^r|) \subset h(U)$. This is accomplished by engulfing the polyhedra P_σ one at a time while keeping $\text{Cl}(H_1(|K^r|) - \bigcup P_\sigma)$ and the previously engulfed P_σ in an open ball at each stage of the engulfing.

Proof of the general case of Theorem 2.4. As in the PL case we may assume that M is r -connected ($1 \leq r \leq n - 3$). Let $f: S^{r+1} \rightarrow M$ be a map. By Lemma 2.3 there exist a map g homotopic to f and a triangulation K of S^{r+1} such that $g(|K^r|)$ is contained in an open n -ball U of M and for each $(r + 1)$ -simplex $\sigma \in K$, $g(\sigma)$ is contained in an open ball U_σ of M . We may further assume that $g|_\sigma: \sigma \rightarrow U_\sigma$ is simplicial relative to the boundary. Now by repeated applications of Lemma 2.2 as in the proof of Lemma 2.3 we may assume that there are a finite number of $(r + 1)$ -simplexes $\sigma_1, \dots, \sigma_m$ in a triangulation of S^{r+1} such that $g(S^{r+1} - \bigcup \text{Int } \sigma_i)$ is contained in an open n -ball of M and each σ_i is PL embedded in an open ball U_i of M . The remainder of the proof is analogous to the PL case.

3. Cantor sets and homotopy connectedness of manifolds. We will now carefully isolate the essential ingredients of the proofs presented in § 2. In § 2 we considered manifolds with the property that each Cantor set was contained in an open ball. We now consider manifolds in which each Cantor set is contained in a certain kind of open set.

Definition 3.1. Let R be a disk with holes. A pair (X, Y) is said to be *disk with holes trivial* if for each map $f: (R, \text{Bd } R) \rightarrow (X, Y)$ there is a map $F: R \rightarrow Y$ such that $F|_{\text{Bd } R} = f|_{\text{Bd } R}$.

Definition 3.2. Suppose U is an open subset of a manifold M . The set U is said to be *trivial at $\text{Fr}_M(U)$* if for every compact set $A \subset U$ there is a set B , $A \subset B \subset U$, which is closed in M such that the pair $(U, U - B)$ is disk with holes trivial.

If the set B can also be chosen compact, then any embedding of U into a manifold of the same dimension will be trivial at $\text{Fr}_M(U)$. Examples of such sets are $M^m \times E^n$ where M^m is a compact m -dimensional manifold and n is at least 3 and other open manifolds with compact spines of codimension at least 3.

Definition 3.3. A manifold M is said to be *Cantor k -connected* if each Cantor set of M lies in a k -connected open set U of M such that U is trivial at $\text{Fr}_M(U)$.

Theorem 3.1. *Suppose M is an n -manifold and $k \leq n - 2$. Then M is k -connected if and only if M is Cantor k -connected.*

Proof. If M is k -connected, then M is clearly Cantor k -connected. If M is Cantor k -connected then the proof is the same as the proof of Theorem 2.4 with minor modifications which we now indicate. The compact 0-dimen-

sional set $P(\bigcup C_i \cup B)$ is contained in a k -connected open set U' of M/B such that U' is trivial at $\text{Fr}_{M/B}(U')$. The set $U = P^{-1}(U')$ contains $\bigcup C_i \cup B$ and is trivial at $\text{Fr}_M(U)$. We let W be a closed subset of M which is contained in U such that the pair $(U, U - W)$ is disk with holes trivial. The remainder of the proof is now the same as the proof of Theorem 2.4.

Although Theorem 3.1 required several technical definitions, these definitions need not be reflected in applications of the theorem. For instance, Theorem 3.1 implies that a 5-manifold in which every Cantor set of the manifold is contained in an open subset homeomorphic to $S^2 \times E^3$ is simply connected.

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