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ON A CONJECTURE OF GROSS

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ABSTRACT. Gross' conjecture about the generalized Laplacian is proved as a consequence of the main theorem.

1. **Gross' conjecture.** The generalized Laplacian of a Borel measurable function f in an abstract Wiener space (H, B) is defined by

$$\Delta f(x) = 2 \lim_{r \downarrow 0} \{E[\tau_x^{(r)}]\}^{-1} \{E[f(x + W(\tau_x^{(r)}))] - f(x)\},$$

where W is a Wiener process in B starting at the origin and $\tau_x^{(r)}$ is the first exit time for $x + W$ from the open ball of radius r in B with center x. Let A be a bounded operator from B to B^* and $u(x) = \frac{1}{2}\langle Ax, x\rangle$. Under the assumption that the B norm $\|\cdot\|$ is twice continuously B-differentiable away from the origin and that the second Fréchet B-derivative is bounded on the annulus $1 \leq \|x\| \leq 2$, Gross [2, p. 148] showed that $\Delta u(0) = \operatorname{trace}(A|H)$. Then he conjectured that the conclusion remains true without any differentiability assumption on the B norm or any assumption concerning the existence of smooth functions on B with bounded support. The purpose of this note is to prove this conjecture. In order to use the results in [3], we assume that there exists a sequence of finite dimensional projections with range in B^* converging strongly to the identity both in B and in H.

2. **Main theorem.** We use the same notation as in [3].

Theorem 1. *Let f be a twice continuously H-differentiable function in B such that $D^2 f(x) \in \mathfrak{B}_1(H, H)$, the Banach space of trace class operators of H, for all x in B and $D^2 f$ is continuous from B to $\mathfrak{B}_1(H, H)$. Let τ be a stopping time (w.r.t. W) such that $\tau < \infty$ a.s. and $\int_1^\infty E[1_{t \leq \tau} |Df(x_0 + W(t))|^2]\, dt < \infty$. Then*

$$E[f(x_0 + W(\tau))] = f(x_0) + E\left[\int_0^\tau \frac{1}{2} \operatorname{trace} D^2 f(x_0 + W(t))\, dt\right].$$

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Proof. From Ito's formula [3, Theorem 4.1]

$$f(x_0 + W(\tau)) = f(x_0) + \int_0^\tau (Df(x_0 + W(t)),\ dW(t))$$
$$+ \int_0^\tau \frac{1}{2} \operatorname{trace} D^2 f(x_0 + W(t))\, dt.$$

Upon taking expectations, we obviously obtain the conclusion provided $E[\int_0^\tau (Df(x_0 + W(t)),\ dW(t))] = 0$. This follows from

Lemma 1. *Let τ be a stopping time and $\sigma(t)$ an H-valued nonanticipating process. Assume $\tau < \infty$ a.s. and $\int_1^\infty E[1_{t\le\tau}|\sigma(t)|^2]\,dt < \infty$. Then $E[\int_0^\tau (\sigma(t),\ dW(t))] = 0$.*

Proof. The proof of [4, §2.3(4)] can be easily modified to show that

$$\int_0^\tau (\sigma(t),\ dW(t)) = \int_0^\infty (1_{t\le\tau}\sigma(t),\ dW(t)).$$

But

$$E\left[\int_n^m (1_{t\le\tau}\sigma(t),\ dW(t))\right]^2 = E\left[\int_n^m |1_{t\le\tau}\sigma(t)|^2\,dt\right]$$
$$= \int_n^m E[1_{t\le\tau}|\sigma(t)|^2]\,dt \longrightarrow 0 \quad \text{as } n,\ m \longrightarrow \infty.$$

This gives the conclusion since $E[\int_0^n (\sigma(t),\ dW(t))] = 0$ for any $n \ge 0$.

Corollary 1. *Let A be a bounded operator from B to B^* and $u(x) = \frac{1}{2}\langle Ax,\ x\rangle$. Then $\Delta u(x) = \operatorname{trace}(A|H)$ for all x in B.*

Proof. It is easy to see that $\|(A|H)^*(h)\|_{B^*} \le \|A\|_{B,B^*}\|h\|$. Hence $(A|H)^*$ extends uniquely to a bounded operator from B to B^*. Obviously, $Du(x) = \frac{1}{2}\{Ax + (A|H)^*x\}$ and $D^2u(x) = \frac{1}{2}\{(A|H) + (A|H)^*\}$. It is well known that $(A|H)$ is a trace class operator of H, hence so is $(A|H)^*$. Let $a = \|A\|_{B,H}$ and $b = \|(A|H)^*\|_{B,H}$. Then

$$E[1_{t\le\tau_x^{(r)}}|Du(x + W(t))|^2] \le (a^2 + b^2)E[1_{t\le\tau_x^{(r)}}\|x + W(t)\|^2]$$
$$\le (a^2 + b^2)r^2 E[1_{t<\tau_x^{(r)}}] = (a^2 + b^2)r^2 P(\tau_x^{(r)} \ge t).$$

But $\{\omega;\ \tau_x^{(r)}(\omega) \ge t\} \subset \{\omega;\ \|W(t)\| \le r\}$. Hence,

$$E[1_{t \le \tau_x^{(r)}} |Du(x + W(t))|^2] \le (a^2 + b^2) r^2 P(\|W(t)\| \le r)$$
$$= (a^2 + b^2) r^2 p_t(S_r),$$

where $S_r = \{y; \|y\| \le r\}$ and p_t is the Wiener measure.

Now, $\int_1^\infty p_t(S_r)\,dt \le \int_0^\infty p_t(S_r)\,dt = G(S_r)$, which is finite by [2, Remark 3.5, p. 147]. Therefore, u satisfies the assumption in Theorem 1 for all x in B and we have

$$E[u(x + W(\tau_x^{(r)}))] - u(x) = \frac{1}{2} E\left[\int_0^{\tau_x^{(r)}} \operatorname{trace}(A|H)\,dt\right] = \frac{1}{2} \operatorname{trace}(A|H) E[\tau_x^{(r)}].$$

It follows obviously that $\Delta u(x) = \operatorname{trace}(A|H)$ for all x in B.

Theorem 2. *Let g be a twice continuously H-differentiable function in an open subset U of B such that $D^2 g$ is continuous from U to $\mathcal{B}_1(H, H)$. Let V be an open subset of U with positive B-distance from the complement of U. Let τ be a stopping time such that*

(a) *almost surely, $\tau < \infty$ and $x_0 + W(t) \in V$ for all $0 \le t \le \tau$, and*

(b) $\int_1^\infty E[1_{t \le \tau} |Dg(x_0 + W(t))|^2]\,dt < \infty$.

Then

$$E[g(x_0 + W(\tau))] = g(x_0) + E\left[\int_0^\tau \frac{1}{2} \operatorname{trace} D^2 g(x_0 + W(t))\,dt\right].$$

Proof. By [1, Lemma 5.4], there exists a twice H-differentiable function ϕ from B to $[0, 1]$ such that (i) $\phi \equiv 1$ in V, (ii) the support of ϕ has positive B-distance from the complement of U and (iii) $D^2\phi$ is continuous from B to $\mathcal{B}_1(H, H)$. This theorem follows by applying Theorem 1 to the function $f(x) = \phi(x) g(x)$.

Corollary 2. *Let g be a twice continuously H-differentiable function in an open subset U of B such that $D^2 g$ is continuous from U to $\mathcal{B}_1(H, H)$. Then*

$$\Delta g(x) = \operatorname{trace} D^2 g(x), \qquad x \in U.$$

Proof. Let $x \in U$. Take V to be a small ball around x. Then

$$\int_1^\infty E[1_{t \le \tau_x^{(r)}} |Dg(x_0 + W(t))|^2]\,dt < \infty$$

in Theorem 2 is satisfied by the same argument as in the proof of Corollary 1. Hence

$$E[g(x + W(\tau_x^{(r)}))] - g(x) = E\left[\int_0^{\tau_x^{(r)}} \frac{1}{2} \operatorname{trace} D^2 g(x + W(t))\, dt\right].$$

The desired conclusion follows from the continuity of $D^2 g$.

3. **General theorem.** Without much extra effort, Theorem 1 can be generalized to diffusion processes. Let $X(t)$ be the solution of the following stochastic integral equation:

$$X(t) = x + \int_0^t A(X(s))\, dW(s) + \int_0^t \sigma(X(s))\, ds,$$

where A and σ are given as in [3, Theorem 5.1].

Theorem 3. *Let f be a twice continuously H-differentiable function in B such that $D^2 f(x) \in \mathfrak{B}_1(H, H)$ for all x in B and $D^2 f$ is continuous from B to $\mathfrak{B}_1(H, H)$. Let τ be a stopping time such that $\tau < \infty$ a.s. and $\int_1^\infty E[1_{t \le \tau} |A^*(X_t) Df(X_t)|^2]\, dt < \infty$. Then*

$$E[f(X(\tau))] = f(x) + E\left[\int_0^\tau \left\{\langle Df(X(t)), \sigma(X(t))\rangle + \frac{1}{2} \operatorname{trace} A^*(X(t)) D^2 f(X(t)) A(X(t))\right\} dt\right].$$

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