ON A CONJECTURE OF GROSS

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ABSTRACT. Gross' conjecture about the generalized Laplacian is proved as a consequence of the main theorem.

1. Gross' conjecture. The generalized Laplacian of a Borel measurable function $f$ in an abstract Wiener space $(H, B)$ is defined by

$$
\Delta f(x) = 2 \lim_{r \downarrow 0} \frac{1}{r^2} \left[ E[f(x + W(r))] - f(x) \right],
$$

where $W$ is a Wiener process in $B$ starting at the origin and $r^{(r)}_x$ is the first exit time for $x + W$ from the open ball of radius $r$ in $B$ with center $x$. Let $A$ be a bounded operator from $B$ to $B^*$ and $u(x) = \frac{1}{2} \langle Ax, x \rangle$. Under the assumption that the $B$ norm $\|\cdot\|$ is twice continuously $B$-differentiable away from the origin and that the second Fréchet $B$-derivative is bounded on the annulus $1 < \|x\| < 2$, Gross [2, p. 148] showed that $\Delta u(0) = \text{trace}(A^*H)$. Then he conjectured that the conclusion remains true without any differentiability assumption on the $B$ norm or any assumption concerning the existence of smooth functions on $B$ with bounded support. The purpose of this note is to prove this conjecture. In order to use the results in [3], we assume that there exists a sequence of finite dimensional projections with range in $B$ converging strongly to the identity both in $B$ and in $H$.

2. Main theorem. We use the same notation as in [3].

Theorem 1. Let $f$ be a twice continuously $H$-differentiable function in $B$ such that $D^2f(x) \in \mathcal{B}_1(H, H)$, the Banach space of trace class operators of $H$, for all $x$ in $B$ and $D^2f$ is continuous from $B$ to $\mathcal{B}_1(H, H)$. Let $\tau$ be a stopping time (w.r.t. $W$) such that $\tau < \infty$ a.s. and $\int_1^\infty E[1_{t<\tau} |Df(x_0 + W(t))|^2] dt < \infty$. Then

$$
E[f(x_0 + W(\tau))] = f(x_0) + E \left[ \int_0^\tau \frac{1}{2} \text{trace} D^2f(x_0 + W(t)) dt \right].
$$

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Proof. From Ito's formula [3, Theorem 4.1]

\[
\int_{x_0}^{x_0 + W(t)} = \int_{x_0}^{t} (Df(x_0 + W(t)), dW(t)) + \int_{0}^{t} \frac{1}{2} \text{trace } D^2 f(x_0 + W(t)) dt.
\]

Upon taking expectations, we obviously obtain the conclusion provided
\[
E[\int_{0}^{t} (Df(x_0 + W(t)), dW(t))] = 0.
\]

This follows from

Lemma 1. Let \( \tau \) be a stopping time and \( \sigma(t) \) an \( H \)-valued nonanticipating process. Assume \( \tau < \infty \) a.s. and \( \int_{1}^{\infty} E[1_{t \leq \tau} |\sigma(t)|^2] dt < \infty \). Then
\[
E[\int_{0}^{\tau} (\sigma(t), dW(t))] = 0.
\]

Proof. The proof of [4, §2.3(4)] can be easily modified to show that
\[
E[\int_{n}^{\tau} (\sigma(t), dW(t))] = 0
\]

This gives the conclusion since \( E[\int_{0}^{\tau} (\sigma(t), dW(t))] = 0 \) for any \( n \geq 0 \).

Corollary 1. Let \( A \) be a bounded operator from \( B \) to \( B^* \) and \( u(x) = \frac{1}{2} \langle Ax, x \rangle \). Then \( \Delta u(x) = \text{trace} (A|H) \) for all \( x \) in \( B \).

Proof. It is easy to see that \( \| (A|H)^*(b) \|_{B^*} \leq \| A \|_{B, B^*} \| b \| \). Hence
\[
(A|H)^* \text{ extends uniquely to a bounded operator from } B \text{ to } B^*. 
\]

But \( \{\omega; \tau^{(r)}(\omega) \geq t\} \subset \{\omega; \|W(t)\| \leq r\} \). Hence,

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\[ E[1_{t \leq \tau_{x}}|Du(x + W(t))|^{2}] \leq (a^{2} + b^{2})r^{2}P(\|W(t)\| \leq r) \]
\[ = (a^{2} + b^{2})r^{2}p_{t}(S_{r}), \]

where \( S_{r} = \{ y; \| y \| \leq r \} \) and \( p_{t} \) is the Wiener measure.

Now, \( \int_{0}^{\infty} p_{t}(S_{r}) \, dt \leq \int_{0}^{\infty} p_{t}(S_{r}) \, dt = G(S_{r}), \) which is finite by \[2, \text{ Remark 3.5, p. 147}]\). Therefore, \( u \) satisfies the assumption in Theorem 1 for all \( x \) in \( B \) and we have

\[ E[u(x + W(\tau_{x}))] - u(x) = \frac{1}{2} E \left[ \int_{0}^{\tau_{x}} \text{trace} (A|H) \, dt \right] = \frac{1}{2} \text{trace} (A|H)E[\tau_{x}]. \]

It follows obviously that \( \Delta u(x) = \text{trace} (A|H) \) for all \( x \) in \( B \).

**Theorem 2.** Let \( g \) be a twice continuously \( H \)-differentiable function in an open subset \( U \) of \( B \) such that \( D^{2}g \) is continuous from \( U \) to \( \mathbb{B}_{1}(H, H) \). Let \( V \) be an open subset of \( U \) with positive \( B \)-distance from the complement of \( U \). Let \( \tau \) be a stopping time such that

(a) almost surely, \( \tau < \infty \) and \( x_{0} + W(t) \in V \) for all \( 0 \leq t \leq \tau \), and

(b) \( \int_{0}^{\infty} E[1_{t \leq \tau}|Dg(x_{0} + W(t))|^{2}] \, dt < \infty. \)

Then

\[ E[g(x_{0} + W(\tau))] = g(x_{0}) + E \left[ \int_{0}^{\tau} \frac{1}{2} \text{trace} \, D^{2}g(x_{0} + W(t)) \, dt \right]. \]

**Proof.** By \[1, \text{ Lemma 5.4}]\), there exists a twice \( H \)-differentiable function \( \phi \) from \( B \) to \([0, 1]\) such that (i) \( \phi \equiv 1 \) in \( V \), (ii) the support of \( \phi \) has positive \( B \)-distance from the complement of \( U \) and (iii) \( D^{2}\phi \) is continuous from \( B \) to \( \mathbb{B}_{1}(H, H) \). This theorem follows by applying Theorem 1 to the function \( f(x) = \phi(x)g(x) \).

**Corollary 2.** Let \( g \) be a twice continuously \( H \)-differentiable function in an open subset \( U \) of \( B \) such that \( D^{2}g \) is continuous from \( U \) to \( \mathbb{B}_{1}(H, H) \). Then

\[ \Delta g(x) = \text{trace} \, D^{2}g(x), \quad x \in U. \]

**Proof.** Let \( x \in U \). Take \( V \) to be a small ball around \( x \). Then

\[ \int_{1}^{\infty} E[1_{t \leq \tau_{x}}|Dg(x_{0} + W(t))|^{2}] \, dt < \infty \]

in Theorem 2 is satisfied by the same argument as in the proof of Corollary 1. Hence
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\[ E[g(x + W(r^r))] - g(x) = \mathbb{E} \left[ \int_0^{r^r} \frac{1}{2} \text{trace } D^2 g(x + W(t)) \, dt \right]. \]

The desired conclusion follows from the continuity of \( D^2 g \).

3. **General theorem.** Without much extra effort, Theorem 1 can be generalized to diffusion processes. Let \( X(t) \) be the solution of the following stochastic integral equation:

\[ X(t) = x + \int_0^t A(X(s)) \, dW(s) + \int_0^t \sigma(X(s)) \, ds, \]

where \( A \) and \( \sigma \) are given as in [3, Theorem 5.1].

**Theorem 3.** Let \( f \) be a twice continuously \( H \)-differentiable function in \( B \) such that \( D^2 f(x) \in \mathcal{B}_1(H, H) \) for all \( x \) in \( B \) and \( D^2 f \) is continuous from \( B \) to \( \mathcal{B}_1(H, H) \). Let \( \tau \) be a stopping time such that \( \tau < \infty \) a.s. and

\[ \int_1^\infty E[1_{t \leq \tau} |A^*(X_t)Df(X_t)|^2] \, dt < \infty. \]

Then

\[ E[f(X(t))] = f(x) + E \left[ \int_0^\tau \left\{ \langle Df(X(t)), \sigma(X(t)) \rangle \right\} \, dt \right] + \frac{1}{2} \text{trace } A^*(X(t))D^2f(X(t))A(X(t)) \, dt. \]

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