ON A CONJECTURE OF GROSS

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ABSTRACT. Gross’ conjecture about the generalized Laplacian is proved as a consequence of the main theorem.

1. Gross’ conjecture. The generalized Laplacian of a Borel measurable function $f$ in an abstract Wiener space $(H, B)$ is defined by

$$\Delta f(x) = 2 \lim_{r \to 0} \frac{1}{r^2} \{E[f(x + W(r))] - f(x)\},$$

where $W$ is a Wiener process in $B$ starting at the origin and $r_x^{(r)}$ is the first exit time for $x + W$ from the open ball of radius $r$ in $B$ with center $x$. Let $A$ be a bounded operator from $B$ to $B^*$ and $u(x) = \frac{1}{2} \langle Ax, x \rangle$. Under the assumption that the $B$ norm $\| \cdot \|$ is twice continuously $B$-differentiable away from the origin and that the second Fréchet $B$-derivative is bounded on the annulus $1 \leq \|x\| \leq 2$, Gross [2, p. 148] showed that $\Delta u(0) = \text{trace}(A[H])$. Then he conjectured that the conclusion remains true without any differentiability assumption on the $B$ norm or any assumption concerning the existence of smooth functions on $B$ with bounded support. The purpose of this note is to prove this conjecture. In order to use the results in [3], we assume that there exists a sequence of finite dimensional projections with range in $B^*$ converging strongly to the identity both in $B$ and in $H$.

2. Main theorem. We use the same notation as in [3].

Theorem 1. Let $f$ be a twice continuously $H$-differentiable function in $B$ such that $D^2 f(x) \in \mathcal{B}_1^r(H, H)$, the Banach space of trace class operators of $H$, for all $x$ in $B$ and $D^2 f$ is continuous from $B$ to $\mathcal{B}_1^r(H, H)$. Let $\tau$ be a stopping time (w.r.t. $W$) such that $\tau < \infty$ a.s. and $\int_1^\infty E[1_{t \leq \tau}|Df(x_0 + W(t))|^2] \, dt < \infty$. Then

$$E[f(x_0 + W(\tau))] = f(x_0) + E\left[\int_0^\tau \frac{1}{2} \text{trace} D^2 f(x_0 + W(t)) \, dt\right].$$

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Proof. From Ito's formula \[3,\] Theorem 4.1
\[
f(x_0 + W(t)) = f(x_0) + \int_0^T (Df(x_0 + W(t)), dW(t)) + \int_0^T \frac{1}{2} \text{trace } D^2f(x_0 + W(t)) \, dt.
\]

Upon taking expectations, we obviously obtain the conclusion provided \(E[\int_0^T (Df(x_0 + W(t)), dW(t))] = 0.\) This follows from

**Lemma 1.** Let \(\tau\) be a stopping time and \(\sigma(t)\) an \(H\)-valued nonanticipating process. Assume \(\tau < \infty\) a.s. and \(\int_1^\infty E[1_{t \leq \tau} |\sigma(t)|^2] \, dt < \infty.\) Then \(E[\int_0^\tau (\sigma(t), dW(t))] = 0.\)

**Proof.** The proof of \([4, \S 2.3(4)]\) can be easily modified to show that

\[
\int_0^\tau (\sigma(t), dW(t)) = \int_0^\infty (1_{t \leq \tau} \sigma(t), dW(t)).
\]

But

\[
E \left[ \int_n^m (1_{t \leq \tau} \sigma(t), dW(t)) \right]^2 = E \left[ \int_n^m |1_{t \leq \tau} \sigma(t)|^2 \, dt \right] = \int_n^m E[1_{t \leq \tau} |\sigma(t)|^2] \, dt \to 0 \quad \text{as} \quad n, m \to \infty.
\]

This gives the conclusion since \(E[\int_0^n (\sigma(t), dW(t))] = 0 \text{ for any } n \geq 0.\)

**Corollary 1.** Let \(A\) be a bounded operator from \(B\) to \(B^*\) and \(u(x) = \frac{1}{2} \langle Ax, x \rangle.\) Then \(\Delta u(x) = \text{trace} (A|H)\) for all \(x\) in \(B.\)

**Proof.** It is easy to see that \(\| (A|H)^*(b) \|_{B^*} \leq \|A\|_{B,B^*} \|b\|.\) Hence \((A|H)^*\) extends uniquely to a bounded operator from \(B\) to \(B^*.\) Obviously, \(Du(x) = \frac{1}{2} \{ Ax + (A|H)^*x \} \) and \(D^2u(x) = \frac{1}{2} \{ (A|H) + (A|H)^* \}.\) It is well known that \((A|H)\) is a trace class operator of \(H,\) hence so is \((A|H)^*.\) Let \(a = \|A\|_{B,H}\) and \(b = \|(A|H)^*\|_{B,H}.\) Then

\[
E[1_{t \leq \tau(x)}|Du(x + W(t))|^2] \leq (a^2 + b^2)E[1_{t \leq \tau(x)}||x + W(t)||^2]
\]

\[
\leq (a^2 + b^2)r^2E[1_{t \leq \tau_x}] = (a^2 + b^2)r^2P(\tau_x \geq t).
\]

But \(\{\omega; \tau_x(\omega) \geq t\} \subset \{\omega; ||W(t)|| \leq r\}.\) Hence,
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\[ E[1_{t \leq r_x} | Du(x + W(t))|^2] \leq (a^2 + b^2)r^2P(\|W(t)\| \leq r) \]
\[ = (a^2 + b^2)r^2\rho_t(S_r), \]

where \( S_r = \{ y; \| y \| \leq r \} \) and \( \rho_t \) is the Wiener measure.

Now, \( \int_0^\infty \rho_t(S_r) dt \leq \int_0^\infty \rho_t(S_r) dt = G(S_r) \), which is finite by \([2, Remark 3.5, p. 147]\). Therefore, \( u \) satisfies the assumption in Theorem 1 for all \( x \) in \( B \) and we have

\[ E[u(x + W(t))] - u(x) = \frac{1}{2} E \left[ \int_0^{r_x} \text{trace} (A|H) dt \right] = \frac{1}{2} \text{trace} (A|H) E[r_x]. \]

It follows obviously that \( \Delta u(x) = \text{trace} (A|H) \) for all \( x \) in \( B \).

**Theorem 2.** Let \( g \) be a twice continuously \( H \)-differentiable function in an open subset \( U \) of \( B \) such that \( D^2g \) is continuous from \( U \) to \( \mathcal{B}_1(H, H) \). Let \( V \) be an open subset of \( U \) with positive \( B \)-distance from the complement of \( U \). Let \( \tau \) be a stopping time such that

(a) almost surely, \( \tau < \infty \) and \( x_0 + W(t) \in V \) for all \( 0 \leq t \leq \tau \), and
(b) \( \int_0^\infty E[1_{t \leq \tau} |Dg(x_0 + W(t))|^2] dt < \infty. \)

Then

\[ E[g(x_0 + W(\tau))] = g(x_0) + E\left[ \int_0^{\tau} \frac{1}{2} \text{trace} D^2g(x_0 + W(t)) dt \right]. \]

**Proof.** By \([1, Lemma 5.4]\), there exists a twice \( H \)-differentiable function \( \phi \) from \( B \) to \([0, 1]\) such that (i) \( \phi \equiv 1 \) in \( V \), (ii) the support of \( \phi \) has positive \( B \)-distance from the complement of \( U \) and (iii) \( D^2\phi \) is continuous from \( B \) to \( \mathcal{B}_1(H, H) \). This theorem follows by applying Theorem 1 to the function \( f(x) = \phi(x)g(x) \).

**Corollary 2.** Let \( g \) be a twice continuously \( H \)-differentiable function in an open subset \( U \) of \( B \) such that \( D^2g \) is continuous from \( U \) to \( \mathcal{B}_1(H, H) \). Then

\[ \Delta g(x) = \text{trace} D^2g(x), \quad x \in U. \]

**Proof.** Let \( x \in U \). Take \( V \) to be a small ball around \( x \). Then

\[ \int_1^\infty E[1_{t \leq r_x} |Dg(x_0 + W(t))|^2] dt < \infty \]

in Theorem 2 is satisfied by the same argument as in the proof of Corollary 1. Hence
\[ E[g(x + W(t))] - g(x) = E \left[ \int_0^{\tau} \frac{1}{2} \text{trace } D^2g(x + W(t)) \, dt \right]. \]

The desired conclusion follows from the continuity of \( D^2g \).

3. General theorem. Without much extra effort, Theorem 1 can be generalized to diffusion processes. Let \( X(t) \) be the solution of the following stochastic integral equation:

\[ X(t) = x + \int_0^t A(X(s)) \, dW(s) + \int_0^t \sigma(X(s)) \, ds, \]

where \( A \) and \( \sigma \) are given as in [3, Theorem 5.1].

Theorem 3. Let \( f \) be a twice continuously \( H \)-differentiable function in \( B \) such that \( D^2f(x) \in \mathcal{B}_1(H, H) \) for all \( x \) in \( B \) and \( D^2f \) is continuous from \( B \) to \( \mathcal{B}_1(H, H) \). Let \( \tau \) be a stopping time such that \( \tau < \infty \) a.s. and

\[ \int_1^\infty E[1_{t < \tau} |A^*(X_t)Df(X_t)|^2] \, dt < \infty. \]

Then

\[ E[f(X(t))] = f(x) + E \left[ \int_0^{\tau} \left\{ \langle Df(X(t)), \sigma(X(t)) \rangle \right. \right. \]

\[ + \frac{1}{2} \text{trace } A^*(X(t))D^2f(X(t))A(X(t)) \left. \right\} \, dt \right]. \]

REFERENCES


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