MULTIPLIERS AND DUALITY IN $A^*$-ALGEBRAS

BOHDAN J. TOMIUK

ABSTRACT. Let $A$ be an $A^*$-algebra which is a dense $^*$-ideal of a $B^*$-algebra. Let $M_r(A)$ be the algebra of all bounded linear right multipliers on $A$. We obtain several characterizations of duality for $A$ in terms of the weak operator topology on $M_r(A)$ and the embedding of $M_r(A)$ into the conjugate space of a Banach space.

1. Introduction. Let $A$ be a (complex) Banach algebra. L. Máté has shown in [4] that if $A$ has a weak right identity then $M_r(A)$ can be embedded anti-isomorphically into the second conjugate space $A^{**}$ of $A$, when $A^{**}$ is considered as a Banach algebra with Arens product. In [5] he has studied multipliers on Banach algebras which satisfy properties other than having a weak right identity. We state these properties in §3 and call them (P1) and (P2). Since every semisimple Banach algebra (and in particular every $A^*$-algebra) has property (P1), it is really property (P2) that plays an important role in this paper.

Let $A$ be a Banach algebra. Máté [5] has constructed a Banach space $Y'$ whose conjugate space $Y'$ is a Banach algebra under a suitable product. If $A$ has properties (P1) and (P2) then $M_r(A)$ can be embedded anti-isomorphically into $Y'^*$ [5, Theorem 4, p. 232]. In particular, if $A$ is a dual $A^*$-algebra which is a dense two-sided ideal of its completion, then $A$ has property (P2) and $M_r(A)$ is anti-isomorphic to $Y'^*$. The converse also holds; namely, if $A$ is an $A^*$-algebra such that $A$ is a dense $^*$-ideal of a $B^*$-algebra, $A$ has property (P2) and $M_r(A)$ is anti-isomorphic to $Y'^*$, then $A$ is dual.

Let $A$ be an $A^*$-algebra which is a dense $^*$-ideal of a $B^*$-algebra. Let $\tau_r$ be the weak operator topology on $M_r(A)$. Our main result states that $A$ is dual if and only if $A$ has property (P2) and $M_r(A)$ is $\tau_r$-complete.

2. Notation and terminology. All algebras and vector spaces are over the complex field.
Let $A$ be a Banach algebra. A mapping $T$ from $A$ into itself is called a right multiplier if $T(xy) = xT(y)$, for all $x, y \in A$. Let $\mathcal{M}_r(A)$ denote the set of all bounded linear right multipliers on $A$. $\mathcal{M}_r(A)$ is a Banach algebra under the usual operations for operators and the operator bound norm.

For any set $S$ in $A$, let $l(S)$ (resp. $r(S)$) denote the left (resp. right) annihilator of $S$ in $A$, and let $c(S)$ denote the closure of $S$ in $A$. If, for every closed left ideal $I$ and every closed right ideal $R$ of $A$, we have $l(r(I)) = I$ and $r(l(R)) = R$, then $A$ is called a dual algebra.

Let $A$ be an $A^*$-algebra. The auxiliary norm on $A$ will be denoted by $| \cdot |$. If $A$ is dual then $| \cdot |$ is unique and the completion $\mathcal{U}$ of $A$ in this norm is a dual $B^*$-algebra [9, Lemma 5.5, p. 54]. For further details on $A^*$-algebras see [7].

If $X$ is a Banach space then $X^*$ and $X^{**}$ will denote the conjugate and second conjugate spaces of $X$. For $x \in X$, $f \in X^*$ we shall also denote the value $f(x)$ by $(x, f)$. $S(X)$ will denote the closed unit ball of $X$. The canonical map of $X$ into $X^{**}$ will be denoted by $\pi$.

Let $X$ be a Banach space and let $B$ be an algebra of bounded linear operators on $X$ into $X$. The weak operator topology on $B$ is the topology on $B$ generated by the seminorms $T \rightarrow \|T(x), f\|$, $x \in X$, $f \in X^*$. Under this topology $B$ is a locally convex topological vector space in which multiplication is separately continuous. For any Banach algebra $A$, we shall denote the weak operator topology on $\mathcal{M}_r(A)$ by $\tau_r$.

3. Arens product and modified Arens product. Let $A$ be a Banach algebra. Arens [1] has defined two products on $A^{**}$ which make $A^{**}$ into a Banach algebra. For completeness we sketch one of the Arens products that we shall need. This is done in stages as follows: Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$. Define $f \ast x \in A^*$ by $(f \ast x)y = f(xy)$. Define $F \ast f \in A^*$ by $(F \ast f)x = F(f \ast x)$. Define $F \ast G \in A^{**}$ by $(F \ast G)\mu = F(G \ast f)$. $A^{**}$ is a Banach algebra under the product $F \ast G$ such that the canonical map of $A$ into $A^{**}$ is an isomorphism of $A$ into $(A^{**}, \ast)$. (See also [3].)

Let $Y$ be the linear hull of the set $\{f \ast x : f \in A^* \text{ and } x \in A\}$ and let $Y^\perp = \{F \in A^{**} : F(f \ast x) = 0 \text{ for all } f \in A^* \text{ and } x \in A\}$. Let $Y'$ be the linear hull of the set

$$b = \sum_{k=1}^{\infty} f_k \ast x_k : \sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty, \ f_k \in A^* \text{ and } x_k \in A.$$  

By [5, Theorem 1, p. 229], $Y'$ is a Banach space under the norm
MULTIPLIERS AND DUALITY IN $A^*$-ALGEBRAS

$\|b\|^* = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|\|x_k\| : b = \sum_{k=1}^{\infty} f_k \ast x_k \right\}$

and $\|b\| \leq \|b\|^*$, for all $b \in Y'$. It is easy to see that if $F \in A^{**}$ then the restriction $F|Y'$ of $F$ to $Y'$ belongs to $Y''$ and $\|F| Y''\| \leq \|F\|$. Máté [5] has introduced a product in $Y''$ (called the modified Arens product) which makes $Y''$ into a Banach algebra. This product is defined as follows: Let $x, y \in A$, $h \in Y'$ and $F, G \in Y'^*$. Define $h \ast x \in Y'$ by $(h \ast x)y = h(xy)$ and define $F \ast h \in Y'$ by $(F \ast h)x = F(h \ast x)$. Define $F \ast G \in Y'^*$ by $(F \ast G)h = F(G \ast h)$. Then $Y''$ is a Banach algebra under the product $F \ast G$.

For each $x \in A$, let $F_x \in Y'^*$ be given by $F_x(h) = h(x)$, for all $h \in Y'$. Then the mapping $x \to F_x$ is an algebraic isomorphism of $A$ into $(Y'', \ast)$ [5, Proposition 2, p. 231]. Let $\pi'(A)$ denote the image of $A$ in $Y'''$ by the mapping $x \to F_x$.

Máté [5] has considered Banach algebras $A$ which have the following properties:

(P1) $\pi'(A) \cap Y''' = (0)$.
(P2) $f_k \in A^*$, $x_k \in A$, $\sum_{k=1}^{\infty} \|f_k\|\|x_k\| < \infty$ and $\sum_{k=1}^{\infty} f_k \ast x_k = 0$ implies that $\sum_{k=1}^{\infty} f_k(x_k) = 0$.

Lemma 3.1. Let $A$ be a dual $A^*$-algebra which is a two-sided ideal of its completion $\mathfrak{A}$. Then $A$ has the properties (P1) and (P2).

Proof. That $A$ has property (P1) follows from the fact that $A$ is semi-simple. (See the proof of [5, Lemma 1, p. 229].) Now suppose that there exist $f_k \in A^*$, $x_k \in A$,

$\sum_{k=1}^{\infty} \|f_k\|\|x_k\| < \infty$ and $\sum_{k=1}^{\infty} f_k \ast x_k = 0$.

Let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in $A$. Then

$\sum_{k=1}^{\infty} f_k(x_k e_\alpha) = \sum_{k=1}^{\infty} (f_k \ast x_k)e_\alpha = 0,$

for all $\alpha$. Since every $a \in A$ can be expressed in the form $a = \sum_\alpha a e_\alpha$ [6, Theorem 16, p. 30], it is easy to see that $\sum_{k=1}^{\infty} f_k(x_k) = 0$. Hence $A$ has property (P2).

4. Multipliers on dual $A^*$-algebras. We shall use the notation of the previous sections. Unless otherwise mentioned, $A$ will be a dual $A^*$-algebra.
which is a two-sided ideal of its completion \( \mathcal{U} \). Let \( I \) be the element of \( Y'^* \) given by

\[
l(h) = \sum_{k=1}^{\infty} f_k(x_k),
\]

where \( h = \sum_{k=1}^{\infty} f_k \ast x_k \in Y' \). By Lemma 3.1, \( I \) is uniquely defined. We have \( F \ast I = F \) for all \( F \in Y'^* \) [5, p. 232] and clearly \( \|I\|' = 1 \). By [5, Theorem 2, p. 231], every \( T \in M_r(A) \) has a unique extension \( T' \) to all of \( Y'^* \) with \( \|T'|' \leq \|T\| \). Let \( F^T = T'(l) \), for all \( T \in M_r(A) \). Since \( A \) has properties (P1) and (P2), it follows that the mapping \( T \rightarrow F^T \) is an anti-isomorphism of \( M_r(A) \) into \( Y'^* \) such that \( F_{T(x)} = T'(F_x) = F_x \ast F^T \), for \( x \in A \) and \( T \in M_r(A) \). (See the proof of [5, Theorem 3, p. 231].)

Lemma 4.1. \( \pi'(A) \) is \( \sigma(Y', Y') \)-dense in \( Y' \).

Proof. Since \( Y \) is dense in \( Y' \), Lemma 3.1 implies that \( \{h \in Y' : F_x(h) = 0, \text{ for all } x \in A\} = (0) \). Hence, by the bipolar theorem [8, 1.5, p. 126], \( \pi'(A) \) is \( \sigma(Y'^*, Y') \)-dense in \( Y'^* \).

Lemma 4.2. For every \( T \in M_r(A) \), we have

\[
(T(a), f) = (f \ast a, F^T) \quad (a \in A, f \in A^*).
\]

Proof. Let \( T^* \) denote the conjugate operator of \( T \). Then \( T^*(f \ast a) = T^*(f) \ast a \) [5, p. 231]. Hence

\[
T'(l)(f \ast a) = l(T^*(f \ast a)) = l(T^*(f) \ast a) = (a, T^*(f)) = (T(a), f).
\]

This completes the proof.

Now consider \( A \) as a left Banach \( A \)-module over itself. Then \( A^{**} \) may also be considered as a left Banach \( A \)-module with \( aF = \pi(a) \ast F \), for all \( a \in A \) and \( F \in A^{**} \). Let \( \text{Hom}_A(A, A^{**}) \) be the set of all bounded linear maps \( T \) from \( A \) into \( A^{**} \) such that \( T(ab) = aT(b) \), \( a, b \in A \).

Lemma 4.3. For each \( F \in A^{**} \), the linear map \( T_F : A \rightarrow A^{**} \) given by

\[
(f, T_F(a)) = (f \ast a, F)
\]

belongs to \( \text{Hom}_A(A, A^{**}) \); moreover \( T_F(a) = \pi(a) \ast F \), for all \( a \in A \).

Proof. Since \( \pi(A) \) is a right ideal of \( (A^{**}, \ast) \) [11, Theorem 5.2, p. 830], \( \pi(a) \ast F \in A^{**} \) for all \( a \in A, F \in A^{**} \). But \( F(f \ast a) = (F \ast f)a = (\pi(a) \ast F)f \). Hence \( T_F(a) = \pi(a) \ast F \), for all \( a \in A \); in particular \( T_F(ab) = aT_F(b) \), for all \( a, b \in A \). Clearly \( T_F \) is linear and continuous. Hence \( T_F \in \text{Hom}_A(A, A^{**}) \).
Corollary 4.4. For each $F \in A^{**}$, there is a unique $T \in M_r(A)$ such that $\pi(T(a)) = T_F(a)$, for all $a \in A$.

Proof. This follows from the fact that $M_r(A)$ and $M_r(\pi(A))$ are isometrically isomorphic and that $\pi(a) \ast F \in \pi(A)$ for all $a \in A$ and $F \in A^{**}$.

Lemma 4.5. The mapping $T \rightarrow F_T$ is an isometric anti-isomorphism of $M_r(A)$ onto $(Y^\ast, \ast)$.

Proof. We have already observed that $T \rightarrow F_T$ is an anti-isomorphism of $M_r(A)$ into $Y^\ast$. Now let $F \in Y^\ast$ and let $T_F$ be the bounded linear map on $A$ into $A^{**}$ given by $(f, T_F(a)) = (f \ast a, F)$, for all $a \in A, f \in A^\ast$. By Lemma 4.1, there exists a net $\{x_\alpha\}$ in $A$ such that $F_{x_\alpha}(b) \rightarrow F(b)$, for all $b \in Y'$.

But, by Corollary 4.4, for each $\alpha$, there exists $T_{x_\alpha} \in M_r(A)$ such that $(f, T_{x_\alpha}(a)) = (f \ast a, F_{x_\alpha})$. Then $(f, T_{x_\alpha}(a)) \rightarrow (f, T_F(a))$, for all $a \in A, f \in A^\ast$, and consequently $T_F(ab) = aT_F(b)$, for all $a, b \in A$. Let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in $A$. We have $\Sigma e_\alpha a = a$, for every $a \in A$. Hence $T_F(a) = T_F(\Sigma e_\alpha a) = \Sigma e_\alpha T_F(a)$, and since $e_\alpha T_F(a) \in \pi(A)$, for all $e_\alpha$, it follows that $T_F(a) \in \pi(A)$, for all $a \in A$. Thus there exists a unique $T \in M_r(A)$ such that $\pi(T(a)) = T_F(a)$, for all $a \in A$. It is easy to see that $F = F_T$. Therefore $T \rightarrow F_T$ is onto $Y^\ast$. To see that $F \rightarrow F_T$ is an isometry, we observe that $(T(a), f) = (f \ast a, F_T)$ gives $\|T\| \leq \|F_T\|'$. On the other hand, $\|F_T\|' = \|T'(l)\|' \leq \|T\|' \|l\|' = \|T\|' \leq \|T\|$. (See the proof of [5, Theorem 2, p. 231].) Hence $\|F_T\|' = \|T\|$, for all $T \in M_r(A)$. This completes the proof.

As an application of Lemma 4.5, we have the following:

Theorem 4.6. Let $A$ be a minimal norm ideal of completely continuous operators on Hilbert space $H$. Then $M_r(A)$ is isometrically anti-isomorphic to $L(H)$, the algebra of continuous linear operators on $H$.

Proof. Clearly $A$ is a dense ideal of $LC(H)$, the algebra of compact linear operators on $H$, and from the proof of [9, Lemma 9.1, p. 64] it follows that $A$ is a dual $A^\ast$-algebra. Let $\alpha$ be the given norm on $A$, $\Re$ the ideal of operators on $H$ with finite dimensional range, $t(\cdot)$ and $r(\cdot)$ the trace and the trace norm on $\Re$. For $T \in L(H)$, let

$$\|T\|_\alpha = \sup \left\{ \frac{|t(XT)|}{\alpha(X)} : 0 \neq X \in \Re \right\}$$

and let $A_\alpha = \{T \in L(H) : \|T\|_\alpha < \infty\}$, $A_\alpha$ is a norm ideal, $\Re \subset A_\alpha$, and the map $\phi : f \rightarrow T_f$ is an isometric isomorphism of $A^\ast$ onto $A_\alpha$, where $f(T) = t(TT_f)$.
for all $T \in A$, $f \in A^*$. (See [10, pp. 74-76].) We have $\phi(f \ast T) = T_f T$, $f \in A^*$, and $T \in A$. Hence $\phi$ maps isomorphically $Y$ into $te(H)$, the algebra of trace-class operators on $H$, and, since every $S \in \mathcal{R}$ is of the form $S = SE$ with $E$ an idempotent in $\mathcal{R}$, we have $\phi(Y) \supset \mathcal{R}$. Since the norms $\alpha, \| \cdot \|_{\alpha}$ are uniform crossnorms, $\| \cdot \|'$ is a uniform crossnorm on $\mathcal{R}$ and therefore $\|S\|' \leq \tau(S)$, $S \in \mathcal{R}$. But, for $X, Z \in \mathcal{R}$, we have $\tau(XZ) = \tau(W^{*}XZ) \leq \alpha(W^{*}X)\|Z\|_{\alpha} \leq \alpha(X)\|Z\|_{\alpha}$ [10, Theorem, p. 4], which gives $\tau(S) \leq \|S\|'$, $S \in \mathcal{R}$. Thus $\|S\|' = \tau(S), S \in \mathcal{R}$. Therefore, since $\mathcal{R}$ is dense in $te(H)$ and $Y$ is dense in $Y'$, it follows that $Y'$ is isometrically isomorphic to $te(H)$. An application of [10, Theorem 2, p. 47] and Lemma 4.5 completes the proof.

We are now ready to prove our main result.

Theorem 4.7. Let $A$ be an $A^*$-algebra which is a dense *-ideal of a $B^*$-algebra $\mathcal{A}$. Then the following statements are equivalent:

(i) $A$ is dual.

(ii) $M_r(A)$ is $r$-complete and $A$ has property (P2).

Proof. (i) $\Rightarrow$ (ii). Suppose $A$ is dual. Then, by Lemma 3.1, $A$ has property (P2) and, by Lemma 4.5, the mapping $T \rightarrow F_T$ is an isometric anti-isomorphism of $M_r(A)$ onto $Y'^*$. Since $S(Y'^*)$ is $w^*$-compact, it easily follows that $S(M_r(A))$ is $r$-compact and hence that $M_r(A)$ is $r$-complete.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Since $A$ has property (P2), the mapping $T \rightarrow F_T$ is an isometric anti-isomorphism of $M_r(A)$ into $Y'^*$. Since $A$ also has property (P1), Lemma 4.1 implies that $\pi'(A)$ is $w^*$-dense in $Y'^*$. For each $x \in A$, let $T_x$ be the right multiplication operator on $A$ given by $x$. The mapping $T \rightarrow F_T$ takes $T_x$ into $F_x$. Thus the image of $M_r(A)$ by the map $T \rightarrow F_T$ is $w^*$-dense in $Y'^*$. Since $M_r(A)$ is $r$-complete, it follows that the map $T \rightarrow F_T$ is onto $Y'^*$. In particular, $S(M_r(A))$ is $r$-compact. Now let $S$ be a bounded set in $A$ and let $T_s$ be the right multiplication operator on $A$ given by $s$, $s \in S$. Let $x \in A$. Since $S(M_r(A))$ is $r$-compact, the weak closure of the set $\{T_s(x) : s \in S\}$ in $A$ is weakly compact. But $\{T_s(x) : s \in S\} = \{xs : s \in S\}$. Hence the left multiplication operator on $A$ given by $x$ is weakly completely continuous (w.c.c.), for every $x \in A$. Thus $A$ is a left w.c.c. $A^*$-algebra. Since the involution in $A$ is continuous, it is also right w.c.c. and consequently a w.c.c. $A^*$-algebra. Hence $\mathcal{A}$ is w.c.c. [6, Lemma 9, p. 29] and therefore dual [6, Theorem 6, p. 21]. Let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in $A$ (and in $\mathcal{A}$). Let $T_\alpha$ be the right multiplication operator on $A$ given by $e_\alpha$. For distinct $e_{\alpha_1}, \ldots, e_{\alpha_n}$ form the sum $T_{\alpha_1} + \cdots + T_{\alpha_n}$. These sums form a bounded set in $M_r(A)$ since
\[ \| (T_{\alpha_1} + \cdots + T_{\alpha_n})a \| = \| ae_{\alpha_1} + \cdots + ae_{\alpha_n} \| \leq k \| a \| || e_{\alpha_1} + \cdots + e_{\alpha_n} || = k \| a \| , \]

for all \( a \in A \). (Since \( A \) is a dense two-sided ideal of \( \mathfrak{U} \), by [6, Lemma 4, p. 18] there exists a constant \( k > 0 \) such that \( \| xy \| \leq k \| x \| \| y \| \), for all \( x \in A \) and \( y \in \mathfrak{U} \).) Since \( \tilde{S}(M_r(A)) \) is \( r \)-compact, the set \( \{ (T_{\alpha_1} + \cdots + T_{\alpha_n})a \} \) has a weak limit point, say \( a' \in A \). Clearly \( ae_{\alpha} = a'e_{\alpha} \) for all \( \alpha \), so that \( a = a' \).

This shows that the (directed) set \( \{ ae_{\alpha_1} + \cdots + ae_{\alpha_n} \} \) converges weakly to \( a \), for every \( a \in A \). Similarly we can show that if the \( e_{\alpha} \) are confined to any subfamily then \( \{ ae_{\alpha_1} + \cdots + ae_{\alpha_n} \} \) also converges weakly to an element of \( A \). Therefore by Orlicz' Theorem [2, p. 240], \( \Sigma a ae_{\alpha} \) is summable to \( a \) in the norm \( \| \cdot \| \) of \( A \). Thus \( a \in \text{cl}(aA) \), for every \( a \in A \), and consequently, by [6, Theorem 16, p. 30], \( A \) is dual. This completes the proof.

**Corollary 4.8.** Let \( A \) be an \( A^* \)-algebra which is a dense \( \ast \)-ideal of a \( B^* \)-algebra. Then \( A \) is dual if and only if the closed unit ball of \( M_r(A) \) is \( r \)-compact.

**Proof.** This follows easily from the proof above.

**Corollary 4.9.** Let \( A \) be an \( A^* \)-algebra which is a dense \( \ast \)-ideal of a \( B^* \)-algebra. Then \( A \) is dual if and only if \( A \) has property (P2) and the mapping \( T \to F^T \) takes \( M_r(A) \) onto \( Y^\ast \).

**Proof.** If \( A \) has property (P2), the proof of Lemma 4.5 shows that the mapping \( T \to F^T \) is an isometry. Hence if \( T \to F^T \) is onto then \( M_r(A) \) is \( r \)-complete.

**Corollary 4.10.** Let \( A \) be an \( A^* \)-algebra which is a dense \( \ast \)-ideal of a \( B^* \)-algebra. If \( A \) is reflexive, then \( A \) is dual.

**Proof.** If \( A \) is reflexive then \( \tilde{S}(A) \) is weakly compact and hence \( \tilde{S}(M_r(A)) \) is \( r \)-compact.

**Remark.** Theorem 4.6 was originally stated for the algebra \( \tau c(H) \). I am grateful to the referee for pointing out that it is valid for all minimal norm ideals. The proof of Theorem 4.6 given here is a modification of the original proof.

**REFERENCES**


2. S. Banach, *Théorie des opérations linéaires*, Monografie Mat., PWN, Warsaw,
288  B. J. TOMIUK


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO K1N 6N5, CANADA

Current address: Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802