THE DIFFERENTIAL EQUATION $\Delta x = 2H(x_u \wedge x_v)$
WITH VANISHING BOUNDARY VALUES

HENRY C. WENTE

ABSTRACT. If $x(u, v)$ is a solution to the system $\Delta x = 2H(x_u \wedge x_v)$
on a bounded domain $G \subset \mathbb{R}^2$ with finite Dirichlet integral and with $x = 0$
on $\partial G$, then $x \equiv 0$ for simply connected $G$, but for doubly-connected $G$ we
construct nontrivial solutions.

I. Introduction. In this paper we demonstrate the following "curious" result.

Theorem. For $G$ a bounded domain in $\mathbb{R}^2$, let $x(u, v): G \to \mathbb{R}^3$ be a
vector function continuous on $\overline{G}$, twice continuously differentiable on $G$,
whose Dirichlet integral $D(x)$ is finite where

\begin{equation}
D(x) = \iint_G (|x_u|^2 + |x_v|^2) \, du \, dv = \iint_G |\nabla x|^2 \, du \, dv
\end{equation}

and which is a solution to the partial differential equation

\begin{equation}
\Delta x = 2H(x_u \wedge x_v)
\end{equation}

for some constant $H$. If $G$ is simply-connected (without loss of generality
we may assume $G = B$ the unit disk $u^2 + v^2 < 1$), then the only such solution
with $x = 0$ on $\partial G$, the boundary of $G$, is $x \equiv 0$. On the other hand if $G$
is doubly-connected (say an annular region), then there exist nontrivial solutions
to (1.2) which vanish on $\partial G$ for any nonzero constant $H$.

If $x(u, v)$ is a solution of the system (1.2) and furthermore satisfies the
conformality condition

\begin{equation}
|x_u| = |x_v|, \quad (x_u \cdot x_v) = 0
\end{equation}
on $G$, then $x(u, v)$ is a parametric representation of a surface of constant
mean curvature $H$. In recent years many existence theorems for the system

Received by the editors November 5, 1973.

AMS (MOS) subject classifications (1970). Primary 35J50, 35J60, 49B25,

Key words and phrases. Dirichlet integral, constant mean curvature, oriented
volume functional.

Copyright © 1975, American Mathematical Society
(1.2), (1.3) have been published, generally subject to the condition that the boundary values \( x: \partial G \to \mathbb{R}^3 \) are an admissible representation of a given Jordan curve \( \gamma \). Such surfaces are of interest as they give representations of soap bubbles. The first result was proven by E. Heinz [5]. For a relatively complete listing of existence proofs we refer the reader to the paper of S. Hildebrandt and H. Kaul [7].

For example S. Hildebrandt [6] proved the following theorem. Let \( \gamma \) be a Jordan curve lying inside the unit ball \( x^2 + y^2 + z^2 \leq 1 \). Let \( \mathcal{S}(\gamma) \) be the class of vector functions \( x: \overline{B} \to \mathbb{R}^3 \) of class \( C^0(\overline{B}) \cap C^1(B) \), with finite Dirichlet integral, and whose boundary values are a representation of \( \gamma \). Then for \( |H| \leq 1 \), among the set of all members of \( \mathcal{S}(\gamma) \) whose range is contained in the unit ball, there is at least one member \( x_0 \) which minimizes the functional

\[
E_H(x) = D(x) + 4HV(x)
\]

where \( V(x) \) is the oriented volume functional

\[
V(x) = \frac{1}{2} \iint_B x \cdot (x_u \wedge x_v) \, du \, dv.
\]

This \( x_0 \) is a solution to the system (1.2), (1.3).

Note. Existence theorems of this nature have also been carried over to the case of nonconstant \( H \). For a full discussion and list of references, the reader should consult K. Steffen [8].

If instead, one considers fixed boundary data, then one obtains solutions to (1.2) but loses the conformality (1.3). Recently, the author obtained the following existence theorem [10].

Let \( \gamma(u, v) \in C^0(\overline{B}) \cap C^1(B) \) be a vector function with finite Dirichlet integral, and denote by \( \mathcal{S}(\gamma) \) the class of functions \( x \in C^0(\overline{B}) \cap C^1(B) \) with \( D(x) < \infty \) and such that \( x - y = 0 \) on \( \partial B \). Suppose also that for every such \( x \), we have \( A(x) > 0 \) where \( A(x) = \iint |x_u \wedge x_v| \, du \, dv \). Let \( K \) be a given constant. Among the set of all \( x \in \mathcal{S}(\gamma) \) such that the oriented volume enclosed by \( x \) and \( y \), \( V(x) - V(y) = K \), there is an \( x_0 \) which minimizes \( D(x) \). This \( x_0 \) is a solution to the system (1.2) for some constant \( H \).

One can regard \( x_0 \) as representing the shape of an ideal elastic membrane in equilibrium with fixed boundary values and subject to a pressure difference on one side.

II. The uniqueness theorem. Since the differential equation (1.2) is invariant under a conformal change of variables in the parameter domain, we
THE DIFFERENTIAL EQUATION $\Delta x = 2H(x_u \wedge x_v)$

may take our simply-connected region $G$ to be the unit disk $B$.

Let $W_1$ be the Sobolev Hilbert space of vector functions $x: B \to \mathbb{R}^3$ such that $x$ and its first partial derivatives $x_u$ and $x_v$ are square integrable. Let $W_{10}$ be the closure in $W_1$ of the $C_0^\infty(B)$ functions, the $C^\infty$ functions with compact support in $B$. $W_{10}$ is a closed subspace of $W_1$ and on $W_{10}$ an equivalent inner product is given by the Dirichlet inner product

\[(x \cdot y)_D = \iint_B (x_u \cdot y_u + x_v \cdot y_v) \, du \, dv = \iint_B (\nabla x \cdot \nabla y) \, du \, dv.\]

The following result was proven in \cite[p. 323]{9}.

The oriented volume functional $V(x)$ has a unique continuous extension from $C_0^\infty(B)$ to $W_{10}$. On $W_{10}$, $V(x)$ is an analytic functional and we have the isoperimetric inequality

\[|V(x)| \leq (1/3\sqrt{32\pi})[D(x)]^{3/2}.\]

The following assertion follows directly from a regularity result proven in \cite[p. 337]{9}.

Let $x \in W_{10}$ be a critical point for the functional $E_H(x)$ in the space $W_{10}$. This means that

\[E'_H(x, \phi) = 2(x \cdot \phi)_D + 4HV'(x, \phi) = 0\]

for all $\phi \in W_{10}$ where $V'(x, \phi)$ is the derivative of $V(x)$ at $x$. Then $x$ is analytic on $B$, continuous on $\overline{B}$ with $x = 0$ on $\partial B$ and is a solution of the differential equation (1.2) on $B$.

We can then state the uniqueness theorem as follows.

**Uniqueness theorem.** The only critical point for the functional $E_H(x)$ in $W_{10}(B)$ is $x = 0$.

**Proof.** Let $x$ be a critical point for $E_H(x)$. Then $x \in C^2(B) \cap C_0^0(\overline{B})$ and is a solution to (1.2) on $B$. We now extend the domain of $x$ to all of $\mathbb{R}^2$ by a reflection, setting

\[y(u, v) = \begin{cases} x(u, v), & (u, v) \in \overline{B}, \\ -x(u/r^2, v/r^2), & (u, v) \not\in \overline{B} \text{ where } u^2 + v^2 = r^2. \end{cases}\]

$y(u, v)$ is then a solution to (1.2) on $\mathbb{R}^2 - \{u^2 + v^2 = 1\}$. We shall now show that $y(u, v)$ is a solution to (1.2) on all of $\mathbb{R}^2$.

**Note.** This same assertion was shown previously by E. Heinz \cite[p. 278]{5} under slightly stronger hypotheses.
Let \( \phi(u, v) \in C^\infty_0(R^2 - \{(0, 0)\}) \). We write \( \phi(u, v) \) as the sum of an even and an odd function relative to the unit circle \( u^2 + v^2 = 1 \). Set

\[
(2.5) \quad \hat{\phi}(u, v) = \phi(u/r^2, v/r^2), \quad u^2 + v^2 = r^2.
\]

We can then write

\[
(2.6) \quad \phi(w) = \frac{\phi(w) + \hat{\phi}(w)}{2} + \frac{\phi(w) - \hat{\phi}(w)}{2} = \phi_e(w) + \phi_o(w)
\]

where \( w = u + iv \). Since \( y(u, v) \) is itself odd, it follows that

\[
(2.7) \quad E_H'(y, \phi_e) = 2(y \cdot \phi_e)_D + 4HV'(y, \phi_e) = 0.
\]

Now let \( \psi = \phi_0 \) restricted to \( \overline{B} \). Then \( \psi \in W_1(B) \cap C^0(\overline{B}) \) and has zero boundary values. Therefore \( \psi \in W_1(B) \). Since \( y \) and \( \phi_0 \) are both odd with respect to \( B \), we have

\[
(2.8) \quad (y \cdot \phi_0)_D = 2(x \cdot \psi)_D, \quad V'(y, \phi_0) = 2V'(x, \psi),
\]

and hence

\[
(2.9) \quad E_H'(y, \phi) = E_H'(y, \phi_0) + E_H'(y, \phi_e) = 2E_H'(x, \psi) + 0 = 0
\]

as \( x \) is a critical point for \( E_H(x) \) on \( W_1(B) \). By the regularity theorem it follows that \( y(u, v) \) is analytic on \( R^2 \) and is a solution to (1.2). Furthermore we have \( D(y) = 2D(x) < \infty \).

We now consider the "conformal measure" function

\[
(2.10) \quad F(w) = (|y_u|^2 - |y_v|^2) - 2i(y_u \cdot y_v) \quad (\text{Courant [2, p. 97]}).
\]

A direct computation shows that \( F(w) \) is complex analytic for any solution of (1.2). Thus for \( y(w) \), \( F(w) \) is an entire function. But we have \( |F(w)| \leq |y_u|^2 + |y_v|^2 \). Therefore

\[
\iint |F(w)| \, du \, dv \leq D(y) = 2D(x) < \infty.
\]

The only entire function in \( L^1(R^2) \) is \( F(w) \equiv 0 \). This means that \( y(u, v) \) satisfies the conformality condition (1.3) as well.

However, by a result of P. Hartman and A. Wintner [4] (for a simple proof applicable here see [3, p. 279]) any solution to the system (1.2), (1.3) other than the constant function possesses only isolated branch points (i.e. points where \( |x_u| = |x_v| = 0 \)). On \( \partial B \) we have \( y \equiv 0 \). This implies that the entire circle \( \partial B \) consists of branch points which are not isolated. Therefore \( y(u, v) \) is constant, and as \( x \in W_1(B) \) we have \( y \equiv 0 \). Q.E.D.
Remark. This result has the following implication as I found out in a discussion with S. Hildebrandt while visiting the University of Bonn.

On $W_{10}$, $x = 0$ is a strong local minimum for the functional $E_H(x)$. From the inequality (2.2) it follows that if $|x|_D = |D(x)|^{1/2} < 3\sqrt{2\pi/8|H|}$, then $E_H(x) > \frac{1}{2} D(x)$. On the other hand, if $H \neq 0$, $E_H(x)$ takes on arbitrarily large negative values. One might hope (applying a minimax argument) that there should exist unstable critical points for $E_H(x)$ on $W_{10}(B)$. Our result shows that this is not the case.

III. Nonuniqueness on the annulus. Since any solution to the differential equation (1.2) is invariant under a conformal transformation of the parameter domain, it follows that any solution to (1.2) on an annular region bounded by the circles $r = 1$ and $r = R_0 > 1$ and vanishing on the boundary, is equivalent to finding a solution $x(w)$ on the infinite vertical strip $0 \leq u \leq 2a$ ($2a = \ln R_0$) which is periodic in $v$ of period $2\pi$ and vanishes on the lines $u = 0$, $u = 2a$. The conformal transformation involved is $T(w) = e^w$. We shall now construct a nontrivial solution on the infinite vertical strip.

We first note that if $x(w)$ is such a solution, then as in §II, one can extend $x(w)$ to a solution on all of $\mathbb{R}^2$ by repeated reflections about the lines $u = 2na$. $x(w)$ will be doubly periodic, of period $4a$ in $u$ and period $2\pi$ in $v$. Therefore, the conformal measuring function $F(w)$ (2.10) is an entire doubly periodic function and so must be a constant. If $x(w)$ is nontrivial we can immediately infer that $F(w) \neq 0$ since otherwise $x(w)$ would be conformal and the lines $u = 2na$ would be lines of branch points.

We shall construct $x(w)$ as a surface of revolution about the z-axis with the form

$$x(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

with the boundary conditions

$$f(0) = g(0) = f(2a) = g(2a) = 0.$$

One straightforwardly concludes that $x(u, v)$ is a solution to the differential equation (1.2) if and only if

$$(a) \ f'' - f = -2Hf'g', \quad (b) \ g'' = 2Hff'. $$

We note here that if $H = 0$, then there are no nontrivial solutions to (3.3) periodic in $u$, as $f'' - f = 0$. In this case, if we also require the mapping to be conformal, solutions to (3.3) will give a representation of the catenary

$$x(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$
Now assume $H \neq 0$. In this case, by multiplying $x(u, v)$ by a suitable scalar factor, we may assume $H = -1$, yielding

$$
(3.4) \quad (a) \quad f'' - f = 2g', \quad (b) \quad g'' = -2f'.
$$

Upon integrating the second of these equations, we get

$$
(3.5) \quad g' = -f^2 + c
$$

for some constant $c$. Substitute this into $(3.4)(a)$ and we get

$$
(3.6) \quad f'' - (2c + 1)f + 2f^3 = 0.
$$

This is the well-known Duffing differential equation in homogeneous form. It has the following integral:

$$
(3.7) \quad (f')^2 - (2c + 1)f^2 + f^4 = \lambda^2
$$

where $\lambda^2 = [f'(0)]^2$ if $f(0) = 0$. We now apply the boundary conditions $(3.2)$, and upon observing that $(3.7)$ is symmetric in $f$ and $f'$, conclude that $f(2a) = f(a) = 0$ if $f$ is periodic of period $4a$, and that

$$
(3.8) \quad ac = \int_0^a f^2(u) \, du
$$

if $g(2a) = 0$. We obtain $(3.8)$ by integrating $(3.5)$ and observing that

$$
\int_0^{2a} f^2(u) \, du = 2\int_0^a f^2(u) \, du
$$

by symmetry. Hence $c > 0$.

Now if $f$ is a solution to $(3.6)$ of period $4a$ and $f(0) = f(4a) = 0$, then an integration of $(3.7)$ yields

$$
(3.9) \quad P(L) = a = \frac{\text{Period}}{4} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{- (2c + 1) + L^2(1 + \sin^2\theta)}}
$$

where $L$ is the amplitude of $f$ which by $(3.7)$ satisfies

$$
(3.10) \quad L^4 - (2c + 1)L^2 = \lambda^2 = [f'(0)]^2
$$

and so $\sqrt{2c + 1} < L < +\infty$ (see [1, p. 154], for example).

Given $c > 0$ we want to choose $L$ so that $(3.8)$ is satisfied. This means

$$
(3.11) \quad \int_0^{\pi/2} \frac{L^2\sin^2\theta \, d\theta}{\sqrt{- (2c + 1) + L^2(1 + \sin^2\theta)}} = cP(L), \quad (2c + 1) < L^2 < \infty,
$$

or $f(L) = cP(L)$. For $P(L)$ we have

$$
(3.12) \quad (a) \quad \text{limit } P(L) = +\infty \quad \text{as } L \to \sqrt{2c + 1},
$$

$$
(b) \quad P(L) \text{ monotonic decreasing},
$$

$$
(c) \quad \text{limit } P(L) = 0 \quad \text{as } L \to +\infty.
$$
For \( \ell(L) \) we have

\begin{align*}
(3.13) \\
(\text{a}) \text{ limit } \ell(L) &= \sqrt{2c+1} \quad \text{as } L \to \sqrt{2c+1}, \\
(\text{b}) \text{ limit } \ell(L) &= +\infty \quad \text{as } L \to +\infty.
\end{align*}

Therefore, there exists at least one value \( \hat{L}, \sqrt{2c+1} < \hat{L} < \infty \), for which (3.11) and hence (3.8) is true. For this choice of \( \hat{L} \), the solution pair \( f(u), g(u) \) of (3.4) vanishes at \( u = 0, u = 2a \) where \( a = P(\hat{L}) \). This gives us our nontrivial solution.

With a more detailed analysis of (3.11), one can show that as \( c \) varies between 0 and \( +\infty \), the corresponding period \( P(L) \) ranges from \( +\infty \) to 0.

Finally we observe that the conformal measuring function \( F(w) \) is given by the integral

\[ F(w) = \int f'^2 + g'^2 - f^2 \quad \text{constant.} \]

If \( f(0) = g(0) = 0 \), then \( f'(0) \neq 0 \) if \( f(u) \) is nontrivial (as \( f \) is a solution to a second order D.E.) and so \( F(w) > 0 \) and our constructed solution is not conformal as we had already observed must be the case.

BIBLIOGRAPHY