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ON CHARACTERIZATIONS OF INNER PRODUCT SPACES

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ABSTRACT. The characterizations of inner product norms given by Tapia [10] make use of a generalized inner product in normed spaces. I shall give simpler proofs and a sharper result using a more geometrical approach.

1. **The generalized inner product of a norm.** Let $f: V \to \mathbf{R}$ be a function defined on the real vector space V. Throughout the paper we shall use the following topology on V: $\lim_{n\to\infty} x_n = x_0$ for $x_n \in V$ iff there is a finite dimensional subspace $V_f \subseteq V$ containing all x_n, x_0 with $x_n \to x_0$ in the conventional topology of V_f.

By definition,

$$f'_+(x)(h) = \lim_{t\to 0+} \frac{f(x+th) - f(x)}{t} \qquad (h, x \in V). \tag{1}$$

The following proposition is elementary (cf. [2, p. 19]):

(α) *Let f be convex; then $f'_+(x)$ (h) exists, and is a convex function of h for every $x \in V$, and for $s > 0$, $f'_+(x)$ $(s \cdot h) = s \cdot f'_+(x)$ (h).*

If $\| \ \|$ is any norm on V, then $\| \ \|$ and

$$g: x \to \|x\|^2/2 \tag{2}$$

are convex functions. By (2), Tapia's generalized inner product exists:

$$(x, h) = g'_+(x)(h) \quad \text{(definition)}. \tag{3}$$

(Note that (3) is a well-known object in differential geometry in case g has good differentiability properties [4], [6], [7], [9].) If $\| \ \|$ is a pre-Hilbert norm, then (3) coincides with the given inner product. In the general case we have

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(4) $$(x, s \cdot h) = (sx, h) = s \cdot (x, h) \quad \text{for } s > 0$$

and

(5) $$(x, x) = \|x\|^2.$$

From

(6) $$f''_+(x)(h_1; h_2) := \lim_{t \to 0+} \frac{1}{t} \{f'_+(x + th_1)(h_2) - f'_+(x)(h_2)\}$$

we obtain, using positive homogeneity,

(7) $$g''_+(x)(x; h) = g'_+(x)(h) = (x, h),$$

and, directly from (6),

(8) $$g''_+(0)(x; h) = (x, h).$$

From the formula

$$g(x + th) - g(x) = \tfrac{1}{2}\{\|x + t \cdot h\| - \|x\|\}\{\|x + t \cdot h\| + \|x\|\}$$

another useful representation is deduced:

(9) $$(x, h) = \|x\| \cdot \lim_{t \to 0+} \frac{1}{t} \{\|x + th\| - \|x\|\}.$$

From (9) the convexity of the generalized inner product in its second variable is most easily seen once more. This yields, by use of the above-mentioned topology,

(β) *The generalized inner product is a continuous function of its second variable.*

Another immediate consequence of (9) is the generalized Cauchy-Schwarz inequality

(10) $$|(x, h)| \le \|x\| \cdot \|h\|.$$

2. **Two lemmas.** The following elementary remarks do not seem to be generally known, though they are useful:

(γ) *A normed space with dimension ≥ 2 is an inner product space iff this holds for every two-dimensional subspace.*

(The lemma (γ), with 3 instead of 2, will be denoted by ($\bar{\gamma}$); ($\bar{\gamma}$) is trivial since in each of the axioms for an inner product there occur at most 3 vectors. Lemma (γ) has been proved by the parallelogram equality of

Jordan-von Neumann. There are geometrical proofs [5] characterizing the ellipsoids with center 0 as the only surfaces which meet each two-dimensional subspace in a central ellipse.) Another elementary consequence of the convexity of g is:

(δ) *In a two-dimensional normed space* g *is differentiable at almost all unit vectors.* (In other words, (x, h) is a linear function of h for $x \in \mathbf{R} \cdot E$, where E contains almost all unit vectors.)

3. **Proofs for Tapia's characterizations.**

Theorem 1. *A normed space is an inner product space iff the generalized inner product* (x, h) *is a linear function of* x.

Proof. In view of (γ) it will be sufficient to consider two-dimensional spaces. By (δ) we may choose a basis $\{e_1, e_2\}$ such that g is differentiable at e_1, e_2. Let $x = \xi_1 e_1 + \xi_2 e_2$, $h \doteq \eta_1 e_1 + \eta_2 e_2$; then the linearity assumption gives

$$(x, h) = \sum \xi_i (e_i, h) = \sum \xi_i \eta_k (e_i, e_k),$$

which is an inner product.

Theorem 2. *A normed space is an inner product space iff* $(x, h) = (h, x)$ *for all* x, h.

Proof. If this symmetry holds, then by (δ) the function (x, h) is linear in x for almost all h, and by (β) it is continuous in h, hence linear everywhere in x. Apply Theorem 1.

Because of (8) the results may be reformulated:

Theorem 3. *A normed space is an inner product space iff* g *is twice Fréchet-differentiable at* 0. *(Then all derivatives of* g *exist at each point.)*

4. **Symmetry of orthogonality (transversality).** The generalized inner product can be traced back to the beginning of our century in connection with the calculus of variations. Blaschke in his theorem [1] on the symmetry of transversality originally made smoothness assumptions.

Theorem 4. *Let* V *be a normed space of dimension* ≥ 3. *Then* V *is an inner product space iff* $(x, h) = 0$ *implies* $(h, x) = 0$. (This does not hold for dimension 2; see [8].)

Proof. Consider a 2-dimensional subspace U of V, and the indicatrix

$I_U = \{x \in U | \; \|x\| = 1\}$. Since this is convex, it has a unit tangent vector at each point: For $x \in I_u$ there exists $h \in I_u$ such that $(x, h) = 0$. Writing $(h, x) = 0$ for $(x, h) = 0$, we get the statement that every unit vector *is* the tangent to I_U at some point. But this means the curve I_U is smooth, for if it had a corner (point with two linearly independent tangent vectors h, h') we could find a unit vector h'' nowhere tangent to I_U, take $h'' = (h - h')/\|h - h'\|$.

The smoothness of the sections I_U implies the smoothness of the whole indicatrix I_V. In particular, for each $x \in V$, the set $\{h | \; (x, h) = 0\}$ is a subspace of V; hence[1]

$$\{h | (h, x) = 0\} \text{ is a subspace of } V. \tag{11}$$

We claim that condition (11) implies V is an inner product space. By observation $(\bar{\gamma})$, that is (γ) with 3 in place of 2, it suffices to show this for V 3-dimensional. In this case, (11) says that under parallel light of direction x the line separating the dark part of the indicatrix surface from the bright part lies in the plane.[2]

By means of elementary geometry [1, pp. 157–158] this gives: Every two-dimensional section I_U is affinely symmetric with respect to each of its diameters. Then I_U is necessarily an ellipse, which is seen either as in [1, pp. 158–159] or without "euclidean" means using Loewner's ellipse of I_U [3]. The rest follows from (γ).

In view of this proof Theorem 4 can be reformulated:

Theorem 5. *Let V be a normed vector space of dimension ≥ 3. Then the following conditions are equivalent*:

(a) *V is an inner product space.*

(b) *For all $x, h \in V$, $(x, h) = 0 \Rightarrow (h, x) = 0$.*

(c) *For all $x \in V$, $\{h \in V | \; (h, x) = 0\}$ is a subspace of V.*

(For 2-dimensional spaces, one has (a) $\Rightarrow$ (b) $\Rightarrow$ (c), but not the reverse implications.)

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[1] I wish to thank the referee for simplifying my original proof up to this point. The reader will easily see that Theorem 4 also holds if the indicatrix is not centrally symmetric, which was included in the author's former proof. The referee also suggested the elegant formulation of Theorem 5.

[2] The following arguments are not new; but former proofs of Theorem 4 always needed smoothness. See also [11, p. 187].

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