\( \theta \)-REFINABILITY AND LOCAL PROPERTIES

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ABSTRACT. If \( Q \) is a property more general than metrizability, we prove several theorems of the general type: A locally \( Q \), \( \theta \)-refinable space is a \( Q \)-space.

1. Introduction. Let \( (Q) \) be a property for a space \( X \). We call a space \( X \) a locally \( (Q) \)-space if each point of the space has an open neighborhood with property \( (Q) \). Smirnov [20] proved that a paracompact, locally metrizable space is metrizable. Ceder [8] proved that a paracompact, locally \( M_i \) space is an \( M_i \)-space for \( i = 1, 2, 3 \). Burke [5] recently showed that a subparacompact, locally developable space is developable.

Throughout this paper, \( n, m \in \mathbb{N} \) and \( \alpha \in A \). A space \( X \) is \( \theta \)-refinable [21] if for every open cover \( U \) of \( X \) there is a sequence \( \{V_n\} \) of open refinements of \( U \) such that if \( x \in X \), there is an \( n(x) \in \mathbb{N} \) such that \( x \) is contained in at most finitely many members of \( \{V_n\} \) (i.e. \( \text{ord}(x, \{V_n\}) < \infty \)). If \( U = \{U_\alpha\} \) is an open cover of \( X \) and \( \{V_n\} \) is a \( \theta \)-refinement of \( U \) we may assume, without loss of generality, that \( V_n = \{V_n(\alpha)\} \) where \( V_n(\alpha) \subseteq U_\alpha \) for each \( \alpha \in A \). Such a collection \( \{V_n\} \) will be called an indexed \( \theta \)-refinement of \( U \).

Clearly every metacompact space is \( \theta \)-refinable and Burke [5] proved that every subparacompact space is \( \theta \)-refinable. We show in Example 4.4 that paracompactness cannot be replaced by subparacompactness, metacompactness or \( \theta \)-refinability in the results of Smirnov and Ceder.

We assume all spaces are \( T_1 \). The positive integers are denoted by \( \mathbb{N} \).

2. Locally semistratifiable spaces. A space \( X \) is a semistratifiable space if for each open set \( U \subseteq X \), there is a collection \( \{U_n\} \) of closed subsets of \( X \) such that \( U = \bigcup_{n=1}^{\infty} U_n \) and if \( U \subseteq V, V \) open, then \( U_n \subseteq V \).

The concept of a semistratifiable space is due to E. Michael and was first studied by Creede [9]. Creede proved that every semistratifiable space is \( \theta \)-refinable and thus \( \theta \)-refinable.
A collection $\mathcal{F}$ of closed subsets of a space $X$ is a *ct-net* for $X$ if for any two distinct points $x, y$, of $X$, there is an $F \in \mathcal{F}$ such that $x \in F$ and $y \not\in F$. A space with a $\sigma$-closure preserving ct-net is called a $\sigma^\#$-space. These definitions were introduced by Siwiec and Nagata in [19].

A space $X$ is a $\beta$-space if for each $x \in X$, there is a sequence $\{g_n(x)\}$ of open neighborhoods of $x$ such that if $x \in g_n(x)$, then $\{x_n\}$ clusters. The first author and Hodel [12] independently defined $\beta$-spaces and proved Theorem 2.1.

**Theorem 2.1.** A space $X$ is a semistratifiable space if and only if $X$ is a $\beta$-space and a $\sigma^\#$-space.

The main result of this section is that a $\theta$-refinable, locally semistratifiable space is semistratifiable. To get this we first obtain the analogous result for $\beta$-spaces and $\sigma^\#$-spaces and then invoke Theorem 2.1.

**Theorem 2.2.** A $\theta$-refinable, locally $\beta$-space is a $\beta$-space.

**Proof.** Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of $X$ by $\beta$-spaces. For each $x \in X$ and for each $\alpha \in A$ such that $x \in U_\alpha$, let $\{g_{n, \alpha}(x)\}$ be a sequence of open neighborhoods of $x$ illustrating that $U_\alpha$ is a $\beta$-space. We may assume $g_{n+1, \alpha}(x) \subseteq g_{n, \alpha}(x)$ for all $n \in \mathbb{N}$. Let $\mathcal{V}_\alpha$ be an indexed $\theta$-refinement of $\mathcal{U}$. For every $x \in X$ and $n \in \mathbb{N}$ there exists an $\alpha_x \in A$ such that $x \in V_n(\alpha_x)$. Let $h_{n,m}(x) = g_{m, \alpha_x}(x) \cap V_n(\alpha_x)$ and put $h_m(x) = \bigcap_{n=1}^m h_{n,m}(x)$. Suppose $x_0 \in h_m(x_m)$. There exists an integer $n_0$ such that $\text{ord}(x, \mathcal{V}_{n_0})$ is finite. For $m > n_0$, $x_0 \in h_{n_0,m}(x_m) \subseteq V_{n_0}(\alpha_{x_m})$. But $\{\alpha_{x_m} : m = 1, 2, \ldots\}$ is a finite set and hence there is an $\alpha \in A$ and a subsequence $N_1 \subseteq N_1 \subseteq \mathbb{N}$ such that $\alpha_{x_j} = \alpha$ for all $j \in N_1$. Thus $x_0 \in h_{n_0,1}(x_j) \subseteq V_j(\alpha_j)$ for all $j \in N_1$. Since $U_\alpha$ is a $\beta$-space, $\{x_j : j \in N_1\}$ clusters and thus the sequence $\{x_m\}$ clusters. Hence $X$ is a $\beta$-space.

In order to establish a theorem for $\sigma^\#$-spaces analogous to Theorem 2.2, we need the following characterization of $\sigma^\#$-spaces due essentially to R. W. Heath.

**Lemma 2.3.** A space $X$ is a $\sigma^\#$-space if and only if for each $x \in X$, there is a sequence $\{g_n(x)\}$ of open neighborhoods of $x$ such that

$$\bigcap_{n=1}^\infty g_n(x) = \{x\}$$

and if $y \in g_n(x)$, then $g_n(y) \subseteq g_n(x)$.

**Theorem 2.4.** A $\theta$-refinable, locally $\sigma^\#$-space $X$ is a $\sigma^\#$-space.

**Proof.** Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of $X$ by $\sigma^\#$-spaces. For each $x \in X$ and for each $\alpha \in A$ such that $x \in U_\alpha$, let $\{g_{n, \alpha}(x)\}$ be a sequence
We first show that if there is a point-finite open refinement of $U$, then $X$ is a $\sigma^\#$-space. Thus let $\mathcal{U} = \{V_\alpha\}$ be an indexed point-finite open refinement of $\mathcal{U}$. For each $x \in X$, let $h_n(x) = \bigcap \{g_{n,\alpha}(x) \cap V_\alpha : x \in V_\alpha\}$. Then it is easy to verify that $\{h_n(x)\}$ satisfies the conditions of Lemma 2.3 for $X$. Thus $X$ is a $\sigma^\#$-space.

Now let $\{H_n\}$ be a $\theta$-refinement of $U$. Let $X_{n,m} = \{x \in X : \text{ord}(x, H_n) \leq m\}$. Then $X_{n,m}$ is a closed subset of $X$ and every point of $X_{n,m}$ is of finite order relative to $H_n$. Since the property of being $\sigma^\#$ is hereditary, $\{U \cap X_{n,m} : U \in \mathcal{U}\}$ is an open cover of $X_{n,m}$ by $\sigma^\#$-spaces. Since $\{H \cap X_{n,m} : H \in \mathcal{H}_n\}$ is a point-finite open refinement of $\{U \cap X_{n,m} : U \in \mathcal{U}\}$, $X_{n,m}$ is a $\sigma^\#$-space. But $X = \bigcup_{n,m} X_{n,m}$. Since the countable union of closed $\sigma^\#$-spaces is clearly $\sigma^\#$, $X$ is a $\sigma^\#$-space.

The following result is an immediate consequence of Theorems 2.1, 2.2 and 2.4.

**Theorem 2.5.** A locally semistratifiable space $X$ is semistratifiable if and only if $X$ is $\theta$-refinable.

Creede [9] has shown that a space $X$ is semimetrizable if and only if $X$ is semistratifiable and first countable. Thus we have the following:

**Theorem 2.6.** A locally semimetrizable space $X$ is semimetrizable if and only if $X$ is $\theta$-refinable.

A class of spaces which simultaneously generalizes $\sigma$-spaces and $M^*$-spaces [13] is the class of $\Sigma$-spaces introduced by Nagami [16]. A space $X$ is a $\Sigma$-space if there is a sequence of locally finite closed covers $\{\mathcal{F}_n\}$ of $X$ such that if $x_n \in \bigcap \{F \in \mathcal{F}_n : x \in F\}$ for some fixed point $x \in X$, then
Michael [14] has pointed out that replacing "σ-locally finite" by "σ-closure preserving" in the definition of Σ-space leads to a strictly larger class of spaces, which are called Σ\(^{#}\)-spaces.

It is unknown if a θ-refinable, locally Σ-space is a Σ-space. In fact, it is not even known if the union of two open Σ-spaces is a Σ-space. However, by Corollary 1.10 and Theorem 3.2 of [16], we have the following partial result.

**Theorem 2.8.** A subparacompact, locally Σ-space is a Σ-space.

On the other hand, using a characterization of Σ\(^{#}\)-spaces given by Nagata [17] (see also [2]) we can obtain the following theorem. The proof is essentially the same as the proof of Theorems 2.2 and 2.4 and is omitted.

**Theorem 2.9.** A θ-refinable, locally Σ\(^{#}\)-space is a Σ\(^{#}\)-space.

### 3. Locally p-spaces and locally w\(\Delta\)-spaces.

By Arhangel'skii [1], a completely regular space \(X\) is called a p-space if there is a sequence \(\{U_n\}\) of open (in \(βX\)) covers of \(X\) such that if \(x \in X\), \(\bigcap_{n=1}^{∞} \text{St}(x, U_n) \subseteq X\). If, in addition, for each \(x \in X\) and \(n \in N\) there exists an \(n(x) \in N\) such that 
\[
\text{St}(x, U_{n(x)}) \subseteq \text{St}(x, U_n),
\]
then \(X\) is called a strict p-space.

A space \(X\) is a w\(\Delta\)-space [4] if there is a sequence \(\{U_n\}\) of open covers of \(X\) such that if \(x \in \text{St}(x, U_n)\), then \(\{x_n\}\) clusters.

Creede [9] introduced the class of quasi-complete spaces which simultaneously generalizes p-spaces and w\(\Delta\)-spaces. A space \(X\) is a quasi-complete space if there is a sequence \(\{U_n\}\) of open covers of \(X\) such that if \(\{x_k: k > n\} \cup \{x\} \subseteq U \in U_n\) for some fixed point \(x \in X\), then \(\{x_n\}\) clusters.

In order to obtain the results of this section we need the following lemma.

**Lemma 3.1 (Burke [6]).** For a completely regular θ-refinable space \(X\), the following are equivalent:

(a) \(X\) is a p-space.
(b) \(X\) is a strict p-space.
(c) \(X\) is a w\(\Delta\)-space.
(d) \(X\) is a quasi-complete space.

Moreover, conditions (c) and (d) are equivalent for any θ-refinable space \(X\).

It should be noted that the equivalence of (a), (b) and (c) is the content of Theorem 1.7 of [6]. The "moreover" is Corollary 3.1.8 of [10].

**Theorem 3.2.** A θ-refinable, locally quasi-complete space \(X\) is quasi-complete.
Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of $X$ by quasi-complete spaces. For each $\alpha \in A$, let $\{\mathcal{U}_{n, \alpha}\}$ be a sequence of open covers of $U_\alpha$ illustrating that $U_\alpha$ is quasi-complete. We may assume $\mathcal{U}_{n+1, \alpha} < \mathcal{U}_{n, \alpha}$ for all $n \in \mathbb{N}$.

Let $\{\mathcal{O}_n\}$ be an indexed $\theta$-refinement of $\mathcal{U}$. Let us put $\mathcal{O}_{n,m} = \{G \cap V_\alpha : G \in \mathcal{O}_{n,m}, \alpha \}$ and $K_k = \bigwedge_{n+m=2}^{k+1} W_{n,m}$. Then for each $k \in \mathbb{N}$, $K_k$ is an open cover of $X$.

Suppose $\{x_i : i > n\} \cup \{x\} \subseteq H_n \in H_n$ for some fixed point $x \in X$. There exists an integer $t_0$ such that $\text{ord}(x, \mathcal{O}_{n_0})$ is finite. For each $k \in \mathbb{N}$, put $S_k = \{x_i : i \geq n_0 + k\}$. Since $S_k \subseteq H_{n_0+k} \in H_n$, it follows that $S_k \subseteq W$ for some $W \in \mathcal{O}_{n_0,k}$. Thus there exists an $\alpha_k \in A$ and a $G_k \in \mathcal{O}_{k, \alpha_k}$ such that $S_k \subseteq G_k \cap V_{\alpha_k}$. But $\{\alpha_k : k \in \mathbb{N}\}$ is an indexed $\theta$-refinement and hence there is an $\alpha \in A$ and a subsequence $N_1 \subseteq N$ such that $\alpha_j = \alpha$ for all $j \in N_1$. Since $S_j = \{x_i : i \geq n_0 + j\} \cup \{x\} \subseteq G_j \in \mathcal{O}_{j, \alpha}$, the sequence $\{x_n\}$ clusters.

The next two results are an immediate consequence of Lemma 3.1 and Theorem 3.2.

**Theorem 3.3.** A $\theta$-refinable, locally $w\Delta$-space is a $w\Delta$-space.

**Theorem 3.4.** A completely regular $\theta$-refinable locally $p$-space (locally strict $p$-space) is a $p$-space (strict $p$-space).

4. Applications and examples.

**Theorem 4.1.** A locally Moore space $X$ is a Moore space if and only if $X$ is $\theta$-refinable.

Proof. A locally Moore space is both locally semistratifiable and locally quasi-complete. Thus, by Theorems 3.2 and 2.5, $X$ is both semistratifiable and quasi-complete. It follows from [9, Theorem 4.6] that $X$ is a Moore space. We note that Theorem 4.1 generalizes, at least for regular spaces, the result of Burke mentioned in the Introduction.

**Theorem 4.2 (Smirnov [20]).** A locally metrizable space $X$ is metrizable if and only if $X$ is paracompact.

Proof. A locally metrizable space is a locally Moore space and hence a Moore space by Theorem 4.1. But a paracompact Moore space is metrizable [3].

The next result follows immediately from Corollary 3.5. For the appropriate definitions the reader is referred to [15].

**Theorem 4.3.** A paracompact, locally $M (M^*, M^\#)$, or $wM$)-space $X$ is an $M (M^*, M^\#, wM)$-space.
Example 4.4. Let $S$ be the space of Example 1 in [11]. The space $S$ is a metacompact Moore space which is locally metrizable, but not metrizable. Thus paracompactness in Theorems 4.2 and 4.3 cannot be replaced by metacompactness, subparacompactness or $\theta$-refinability.

Example 4.5. Let $X$ be the space constructed by Burke in [7]. This is an example of a locally compact, locally metrizable space which is not $\theta$-refinable. This example shows that $\theta$-refinability is necessary in Theorems 2.2, 2.5–2.7, 2.9, 3.3, and subparacompactness is necessary in Theorem 2.8.

REFERENCES


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