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EXPONENTIAL SOLUTIONS OF $y'' + (r - q)y = 0$ AND THE LEAST EIGENVALUE OF HILL'S EQUATION

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ABSTRACT. It is shown that if q is a nonnegative continuous function on $[0, \infty)$ such that for some positive constants A and L,

$$\liminf_{x\to\infty} \int_x^{x+A} q^{1/2}(t)dt > AL,$$

then $y'' + (r - q)y = 0$ has an exponentially increasing solution and an exponentially decreasing solution whenever the uniform norm of the continuous function r satisfies $\|r\|_\infty < [L/(AL+1)]^2$. A refinement of the proof is used to show that for all sufficiently large values of k the least eigenvalue $\lambda(k)$ of the two parameter Hill equation $y'' + (\lambda - kp)y = 0$ satisfies an inequality of the form $\lambda(k) \geq Pk + B_\beta|k|^\beta$ where $P = \min p$ if $k > 0$, $P = \max p$ if $k < 0$, and β is a constant between 0 and 1 that depends on the periodic function p.

1. **Introduction.** If q is a nonnegative continuous function on $[0, \infty)$, then it is well known that the equation

$$z'' - qz = 0 \tag{1}$$

has a positive increasing solution and a positive decreasing solution on $[0, \infty)$. If q is bounded away from 0, then it is not difficult to see that (1) has an exponentially increasing solution and an exponentially decreasing solution. In fact the following weaker condition is known to be sufficient [4].

Theorem 1. *Suppose that*

$$\liminf_{x\to\infty} \left(\int_0^x q^{1/2}(t)dt/x\right) > L > 0.$$

Then there are positive solutions z and z_0 of (1) *such that for any $a > 0$ there exists $b \geq a$ so that*

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(i) $z(x) \geq z(a)e^{L(x-a)}$,

(ii) $z_0(x) \leq z_0(a)e^{L(a-x)}$

for all $x \geq b$.

If z''/z is not assumed to be of constant sign, then the behavior of solutions of (1) is less clear. The first purpose of this paper is to determine conditions sufficient for the equation

$$y'' + (r - q)y = 0 \tag{2}$$

to have solutions like those in Theorem 1. We show that if q is nonnegative and if, for some positive constants A and L,

$$\liminf_{x \to \infty} \int_x^{x+A} q^{1/2}(t)dt > AL,$$

then (2) has such solutions whenever r is a continuous function whose uniform norm satisfies $\|r\|_\infty < [L/(AL+1)]^2$. Since the first inequality does not exclude the possibility that q vanishes on each of a sequence of intervals of bounded length, our result applies to some equations (2) for which

$$\limsup_{x \to \infty} [r(x) - q(x)] > 0.$$

In particular, using the Sturm separation theorem, such equations are nonoscillatory, that is, each solution of (2) has only finitely many zeros on $[0, \infty)$. This does not seem to be a consequence of the standard nonoscillation criteria.

It is clear that the less restrictive hypothesis on q of Theorem 1 is not sufficient for such a conclusion, for it can be satisfied by a function q that vanishes on a sequence of intervals whose lengths are unbounded, and then (2) must be oscillatory for any positive constant r.

We then establish a lower bound for the least eigenvalue of the two parameter Hill equation. Thus, if p is a continuous real-valued periodic function with period 1 and mean value 0 which assumes values near its extreme values only on a relatively small set, we estimate, as a function of k, the smallest real number $\lambda = \lambda(k)$ such that

$$y'' + (\lambda - kp)y = 0 \tag{3}$$

has a periodic solution on $\mathbf{R}$. Our estimate is of the form $\lambda(k) \geq Pk + B_\beta|k|^\beta$ for $|k|$ sufficiently large, where $P = \min p$ for $k > 0$ and $P = \max p$ for $k < 0$. Here β is between 0 and 1 and depends on the rate of growth of $p - \min p$ and $\max p - p$ when these functions are small. The proof is based on Hamel's observation [1, p. 404] that (3) is nonoscillatory if and only if $\lambda \leq \lambda(k)$, and on a suitable variant of the proof of the result on exponential solutions. R. A. Moore [3] obtained upper bounds for $\lambda(k)$ of this type, and also obtained lower bounds for certain examples.

2. **Exponential solutions.** Our result for the equation (2) may be stated as follows.

Theorem 2. *Let q be a nonnegative continuous function on $[0, \infty)$ such that for some positive constants A and L,*

$$\liminf_{x\to\infty} \left(\int_x^{x+A} q^{1/2}(t)\,dt\right) > AL.$$

Let r be any continuous function such that $\|r\|_\infty < [L/(AL + 1)]^2$. Set

$$M = AL^2/(AL + 1) + ([L/(AL + 1)]^2 - \|r\|_\infty)^{1/2}.$$

Then (2) has eventually positive solutions y and y_0 such that for any a greater than some a_0 there exists $b \geq a$ such that

(i) $y(x) \geq y(a)e^{M(x-a)}$ *and*

(ii) $y_0(x) \leq [K/y(a)]e^{M(a-x)}$

for all $x \geq b$.

We remark that as q becomes more regular it is possible to choose a smaller A and thus to permit a relatively larger r. In the extreme case when q is constant, Theorem 2 includes the statement that (2) has exponentially increasing and decreasing solutions whenever $\|r\|_\infty < q$.

It is convenient to begin with two lemmas.

Lemma 1. *Suppose that f and r are continuous functions on $[0, \infty)$ such that for some positive constant k, $\inf f(x) \geq k$ and $\|r\|_\infty < k^2$. Then any solution of*

$$u'' + 2fu' + ru = 0 \tag{4}$$

such that $u(0) > 0$ and $u'(0)/u(0) \geq -K = -k + (k^2 - \|r\|_\infty)^{1/2}$ satisfies $u'(x)/u(x) \geq -K$ for all nonnegative x.

Proof. If $v(x) = u(x)e^{Kx}$ where u satisfies (4), then v satisfies

$$v'' + 2(f - K)v' + (K^2 - 2fK + r)v = 0. \tag{5}$$

Now $K^2 - 2fK + r \leq K^2 - 2kK + \|r\|_\infty = 0$. Hence any solution v of (5) with $v(0) > 0$, $v'(0) > 0$ is increasing. This implies the statement about $u'/u = v'/v - K$.

Lemma 2. *Let q be a nonnegative continuous function on* $[0, \infty)$ *such that*

$$\liminf_{x \to \infty} \left(\int_x^{x+A} q^{1/2}(t)\,dt \right) > AL.$$

Then there exists x_0 *so that any solution* z *of* (1) *with* $z'(x_0)/z(x_0) \geq L$ *satisfies* $z'(x)/z(x) \geq L/(AL + 1)$ *for all* $x \geq x_0$.

Proof. Fix $L' > L$, $B < A$, and x_0 so that $2LB < A(L' - L) < 1$ and $\int_x^{x+A} q^{1/2}(t)dt > AL'$ for all $x \geq x_0$. Let $w = z'/z$. Then w satisfies

$$w' = q - w^2, \qquad w(x_0) \geq L, \tag{6}$$

and we wish to show that $w(x) \geq L/(AL + 1)$ for all $x \geq x_0$. We shall first construct an increasing sequence $\{x_n\}_{n=0}^{\infty}$ such that $B \leq x_{n+1} - x_n \leq A$ and $w(x_n) \geq L$ for each n. Suppose that $x_0, x_1, \cdots, x_n$ have already been chosen. On $[x_n, x_n + A]$, w is positive and so (6) may be rewritten as $w'/w + w = q/w$. From Schwarz' inequality,

$$\left[\int_{x_n}^{x_n+A} q^{1/2}(t)\,dt \right]^2 \leq \int_{x_n}^{x_n+A} w(t)\,dt \int_{x_n}^{x_n+A} q(t)/w(t)\,dt$$

$$= \int_{x_n}^{x_n+A} w(t)\,dt \left[\int_{x_n}^{x_n+A} w(t)\,dt + \log w(x_n + A) - \log w(x_n) \right].$$

Thus either $w(x_n + A) \geq w(x_n)$ or $\int_{x_n}^{x_n+A} w(t)dt > \int_{x_n}^{x_n+A} q^{1/2}(t)dt \geq AL'$. In the first case we may choose $x_{n+1} = x_n + A$. In the second case, suppose that $w(x) < L$ for $x_n + B \leq x \leq x_n + A$. Then

$$\int_{x_n}^{x_n+B} w(t)\,dt = \int_{x_n}^{x_n+A} w(t)\,dt - \int_{x_n+B}^{x_n+A} w(t)\,dt \geq A(L' - L).$$

Thus for some $X \in [x_n, x_n + B]$, $w(X) \geq A(L' - L)/B \geq 2L$. For $x \geq X$, w satisfies the differential inequality $w' \geq -w^2$ and hence $w(x) \geq u(x) = 1/(1/2L + x - X)$, where u is the solution of $u = -u^2$, $u(X) = 2L$. But then $w(x) \geq L$ for $X \leq x \leq X + 1/2L$ which is a contradiction since $X + 1/2L \geq x_n + 1/2L > x_n + B$. Thus in the second case we may choose $x_{n+1} \in [x_n + B, x_n + A]$.

Now on any interval $[x_n, x_{n+1}]$, w satisfies $w' \geq -w^2$, $w(x_n) \geq L$. Since $x_{n+1} - x_n \leq A$, this implies as above that $w(x) \geq (1/L + A)^{-1} = L/(AL + 1)$ for $x_n \leq x \leq x_{n+1}$. Hence $w(x) \geq L/(AL + 1)$ for all $x \geq x_0$, and the proof of Lemma 2 is complete.

We note that since any positive increasing solution z of (1) is a linear combination of a solution satisfying the hypotheses of Lemma 2 and an exponentially decreasing solution whose derivative tends monotonically to 0, it follows from Lemma 2 that any such solution satisfies

$$\liminf_{x\to\infty} \frac{z'(x)}{z(x)} \geq \frac{L}{(AL+1)}$$

whenever the first hypothesis of Lemma 2 is satisfied.

Proof of Theorem 2. Let x_0 be as in Lemma 2. By Lemma 2 there is a positive solution z of (1) such that $z'(x)/z(x) \geq L/(AL+1) = k$ if $x \geq x_0$. Let K be as in Lemma 1, and let y be a solution of (2) such that $y(x_0) > 0$, $y'(x_0)/y(x_0) \geq L - K$. Set $u = y/z$. Then u satisfies $u'' + 2(z'/z)u' + ru = 0$, and $u'(x_0)/u(x_0) \geq -K$. Hence by Lemma 1, $u'/u \geq -K$ on $[x_0, \infty)$ and so for any $a \geq x_0$, $u(x) \geq u(a)e^{-K(x-a)}$ for $x \geq a$. By Theorem 1 there exists $b \geq a$ such that $z(x) \geq z(a)e^{L(x-a)}$ for $x \geq b$. Hence

$$y(x) = z(x)u(x) \geq y(a)e^{(L-K)(x-a)} = y(a)e^{M(x-a)}$$

for $x \geq b$ where M is as in the statement of Theorem 2. This establishes (i).

It follows from (i) and the Sturm separation theorem that (2) is nonoscillatory. Thus there exists a solution y_0 of (2), positive for all x greater than some a_0, such that $y_0^{-1} \notin L^2(a_0, \infty)$ [2, p. 355]. We may assume $a_0 \geq x_0$. To establish (ii) it suffices to show that yy_0 is bounded above, where y is as in the first part of the proof. Consider the Wronskian $y'y_0 - yy_0' = C$. Dividing by yy_0 we obtain

$$y'/y = C/yy_0 + y_0'/y_0. \tag{7}$$

Now $y'/y = z'/z + u'/u \geq k - K = (k^2 - \|r\|_\infty)^{1/2} > 0$ on $[a_0, \infty)$. We assert that $C/yy_0 \geq (k^2 - \|r\|_\infty)^{1/2}$ on the same interval. Suppose that this inequality fails for some $X \geq a_0$. Then $y_0'(X) > 0$ and in fact we must have $y_0'(x) > 0$ for all $x \geq X$, for if X' is the smallest zero of y_0' greater than X then

$$C/yy_0(X') \geq (k^2 - \|r\|_\infty)^{1/2}.$$

On the other hand, yy_0 is increasing on $[X, X']$ so that $C/yy_0'(X') < C/yy_0(X) < (k^2 - \|r\|_\infty)^{1/2}$. Thus either yy_0 is bounded above or y_0 is eventually increasing. In the second case clearly $yy_0 \to \infty$ so that

$$\liminf_{x\to\infty} y_0'(x)/y_0(x) \geq (k^2 - \|r\|_\infty)^{1/2}$$

follows from (7). But this is impossible since $y_0^{-1} \notin L^2(a_0, \infty)$. Hence yy_0 is bounded above and Theorem 2 is proved.

3. **Hill's equation.** We turn now to the two-parameter Hill equation. The essential tool will be a form of Lemma 2 for the equation

$$z'' - kqz = 0 \tag{8}$$

which establishes the rate at which the minimum of $w = z'/z$ increases as a function of k. We note that if q vanishes on an interval $[a, b]$, then $w' = -w^2$ on $[a, b]$ and so $w(b) \leq 1/(b-a)$ for any k and any value of $w(a)$. On the other hand, it is easy to see that if w satisfies a Riccati equation $w' = Q - w^2$ where $Q(x) \geq c > 0$ on an interval $[a, b]$ and if $w(a) \geq \sqrt{c}$, then $w(x) \geq \sqrt{c}$ for all $x \in [a, b]$. Thus if in (8), $q(x) \geq c > 0$ for all x, then the solution z_k of (8) with $z_k(0) = 1$, $z_k'(0) = (kc)^{1/2}$ satisfies $z_k'/z_k \geq (kc)^{1/2}$ on $[0, \infty)$. Thus it is natural to expect that the rate at which the minimum of z_k'/z_k increases as a function of k depends upon the nature of q on the set where q assumes arbitrarily small values. In fact we shall now establish the following result. We use m to denote Lebesgue measure.

Theorem 3. *Let q be a nonnegative continuous function on $[0, \infty)$ such that*

(i) *for some positive constants A and L,*

$$\liminf_{x\to\infty} \left(\int_x^{x+A} q^{1/2}(t)dt\right) \geq AL,$$

(ii) *for some positive constants α, B, c_0, X,*

$$\mu(c) = m\{t \in [x, x+A]: q(t) \leq c\} \leq Bc^\alpha$$

for all $x \geq X$ and all positive $c \leq c_0$.

Then for all $k \geq k_0$ and all $x \geq x_0$, the solution z_k of (8) with $z_k(x_0) = 1$, $z_k'(x_0) = L\sqrt{k}$ satisfies

$$z_k'(x)/z_k(x) \geq Dk^{\alpha/(2\alpha+1)}$$

for all $x \geq x_0$ with $D = (1/2)B^{-1/(2\alpha+1)}$.

Proof. Choose $x_0 \geq X$ as in the proof of Lemma 2. Set $w_k = z_k'/z_k$. Then $w_k' = kq - w^2$, $w_k(x_0) = L\sqrt{k}$. It now follows from the argument in the proof of Lemma 2 that there is an increasing unbounded sequence $\{x_n\}_{n=0}^\infty$ such that $x_{n+1} - x_n \leq A$ and $w_k(x_n) \geq L\sqrt{k}$ for each n. Fix k_1 so that $L\sqrt{k} \geq 2Dk^{\alpha/(2\alpha+1)}$ for all $k \geq k_1$. It now suffices to consider one interval

$[x_n, x_{n+1}]$. Fix $D' < D$. There exists k_2 so that

$$m\{t \in [x_n, x_{n+1}]: kq(t) \le 4D^2 k^{2\alpha/(2\alpha+1)}\}$$

$$= \mu(4D^2 k^{-1/(2\alpha+1)}) \le B(4D^2)^\alpha k^{-\alpha/(2\alpha+1)} \le (1/2D)k^{-\alpha(2\alpha+1)}$$

for all $k \ge k_2$.

Let $\{(a_i, b_i)\}_{i=1}^l$ be disjoint intervals in $[x_n, x_{n+1}]$ so that $a_{i+1} > b_i$ for each i,

$$\sum_{i=1}^{l} (b_i - a_i) < \left(\frac{1}{2D'}\right) k^{-\alpha(2\alpha+1)}$$

and $kq(t) \ge 4D^2 k^{2\alpha/(2\alpha+1)}$ on $[x_n, a_1]$, $[b_l, x_{n+1}]$ and each interval $[b_i, a_{i+1}]$. We can now estimate w_k on $[x_n, x_{n+1}]$. It will be convenient to use the notation

$$K_0 = 2Dk^{\alpha/(2\alpha+1)}, \qquad K_j = \left[K_0^{-1} + \sum_{i=1}^{j} (b_i - a_i)\right]^{-1}, \qquad j = 1, 2, \cdots, l.$$

(a) $w_k(x_n) \ge K_0$ and $(kq(x))^{1/2} \ge K_0$ on $[x_n, a_1]$. Hence $w_k \ge K_0$ on $[x_n, a_1]$.

(b) On $[a_1, b_1]$, $w_k' \ge - w_k^2$. Hence $w_k \ge K_1$ on $[a_1, b_1]$.

(c) On $[b_1, a_2]$ we have again $(kq(x))^{1/2} \ge K_0$. Thus if $w_k(b_1) \ge K_0$, then $w_k \ge K_0$ on $[b_1, a_2]$. Otherwise it is easy to see that $w_k(x) \ge w_k(b_1)$ for $x \in [b_1, a_2]$. In either case $w_k \ge K_1$ on $[b_1, a_2]$.

Next, as in (b), we obtain $w_k \ge K_2$ on $[a_2, b_2]$ and by alternating steps of type (b) and (c) we arrive finally at the conclusion that

$$w_k(x) \ge K_l \ge D' k^{\alpha/(2\alpha+1)} \quad \text{for all } x \in [x_n, x_{n+1}].$$

Since this inequality holds for all $k \ge \max(k_1, k_2)$, all positive integers n, and all $D' < D$, the theorem is proved.

We are now in a position to estimate, as a function of k, the values of λ for which (3) is nonoscillatory, and thereby to obtain a lower bound for the smallest real number $\lambda(k)$ for which (3) has a periodic solution.

Theorem 4. *Let p be a periodic function with period* 1 *and mean value* 0. *Set $M = \max p$ and $N = \min p$.*

(i) *If $m_1(c) = m\{(0, 1): p(x) - N \le c\} \le Bc^\alpha$ for some positive constants B and α and all sufficiently small positive c, then for all $k \ge k_1 > 0$,*

$$\lambda(k) \ge Nk + Ck^{2\alpha/(2\alpha+1)}.$$

(ii) *If* $m_2(c) = m\{(0, 1): M - p(x) \le c\} \le Bc^{\alpha}$ *for some positive constants* B *and* α *and all sufficiently small* c, *then for all* $k \le k_2 < 0$,

$$\lambda(k) \ge Mk + C(-k)^{2\alpha/(2\alpha+1)}.$$

We remark that if $p \in C^3(\mathbf{R})$ and neither $p - N$ nor $M - p$ has any triple zeros, then it follows from Taylor's theorem that the hypotheses in (i) and (ii) are satisfied with $\alpha = \frac{1}{2}$. If p is piecewise linear and not equal to M or N on any interval, then the hypotheses in (i) and (ii) are satisfied with $\alpha = 1$.

We note also, as mentioned above, that R. A. Moore [3] obtained upper bounds for $\lambda(k)$ of this form under hypotheses which imply our hypotheses with the inequalities reversed. Thus the exponent $2\alpha/(2\alpha + 1)$ cannot in general be improved.

Proof. Set $q(x) = p(x) - N$. Then the function μ defined in the statement of Theorem 3 satisfies $\mu(c) = m_1(c) \le Bc^{\alpha}$ for all small c. Hence by the theorem there exists k_1 so that the solution z_k of (8) with $z_k(0) = 1$, $z_k'(0) = \sqrt{k}\int_0^1 q^{1/2}(t)dt$ satisfies $z_k'/z_k \ge Dk^{\alpha/(2\alpha+1)}$ on $[0, \infty)$ for all $k \ge k_1$. By Lemma 1 $y'' + (r - kq)y = 0$ has a positive solution on $[0, \infty)$ whenever $\|r\|_{\infty} < D^2k^{2\alpha/(2\alpha+1)}$. Thus (3) has a positive solution on $[0, \infty)$ whenever $\lambda \le Nk + D^2k^{2\alpha/(2\alpha+1)}$. From the change of independent variable $t = -x$ it is clear that (3) has a positive solution on $(-\infty, 0]$ for the same values of λ. Hence, by the Sturm separation theorem, (3) is nonoscillatory for all such λ. Thus by the remark of Hamel mentioned in the Introduction, $\lambda(k) \ge Nk + Ck^{2\alpha/(2\alpha+1)}$. This proves (i). The proof of (ii) is similar.

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