CRITERIA FOR ABSOLUTE CONVERGENCE OF FOURIER SERIES

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ABSTRACT. Let $f \in L^1(T)$. Define $f_a$ by $f_a(x) = f(x + a)$. Then the Fourier series of $f$ is absolutely convergent if and only if there exists a Lebesgue point $a$ for $f$ such that both sequences $\langle \text{Re } \hat{f}_a(n) \rangle_{n \in \mathbb{Z}}$ and $\langle \text{Im } \hat{f}_a(n) \rangle_{n \in \mathbb{Z}}$ belong to $l^1$. The theorem remains true if the sentence "there exists a Lebesgue point $a$ for $f$" is replaced by "there is $a \in \mathbb{R}$ such that $f$ is essentially bounded in some neighborhood of $a$".

1. Introduction. One of the primary objectives in the theory of Fourier series is the study of the class $\mathcal{C}$, that is the class of all Lebesgue integrable complex-valued functions on the circle $T$ (the additive group of the reals modulo $2\pi$) whose Fourier series are absolutely convergent. We denote by $A$ the collection of all continuous complex-valued functions on $T$ with absolutely convergent Fourier series. Clearly $A \subset \mathcal{C}$, and every function in $\mathcal{C}$ is equal almost everywhere to a function in $A$. An approach to studying $\mathcal{C}$ or $A$ (for which the reader is referred to [1]) has concentrated attention on seeking conditions on a function $f$, which are sufficient or necessary, that ensure that $f \in \mathcal{C}$. It seems that there is no complete solution to the problem of characterizing the elements of $\mathcal{C}$ directly in terms of their functional values. The following theorem is well known [1, p. 9]: Every continuous function on $T$ with nonnegative Fourier coefficients belongs to $A$. Every function in $A$ is a linear combination of continuous functions on $T$ with nonnegative Fourier coefficients.

In this paper we first give an easy generalization of this theorem by requiring that the Fourier coefficients be confined in a certain region of the complex plane instead of lying all on the positive $x$-axis. This is the content of Theorem 1. Next we prove our main results given by Theorems 2 and 4 of §3. Each of these theorems provides a necessary and sufficient condition for a function to be in the class $\mathcal{C}$.

Received by the editors April 24, 1974.


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2. Notation. \( \mathbb{R} \) is the real line. \( L^1(\mathbb{T}) \) is the space of all Lebesgue integrable complex-valued functions on \( \mathbb{T} \). For each \( f \in L^1(\mathbb{T}) \) the numbers
\[
\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} dt \quad (n \in \mathbb{Z})
\]
are the Fourier coefficients of \( f \), where \( \mathbb{Z} \) is the group of integers. As usual we call \( \sum_{n \in \mathbb{Z}} \hat{f}(n)e^{int} \) the Fourier series of \( f \). Let \( f \) be Lebesgue integrable on an interval \([a, b) \subset \mathbb{R}\) and let \( x \in (a, b) \). Then \( x \) is called a Lebesgue point for \( f \) if and only if
\[
\lim_{h \to 0^+} \frac{1}{h} \int_{0}^{h} |f(x + t) + f(x - t) - 2f(x)| dt = 0.
\]
The set of all Lebesgue points for \( f \) is called the Lebesgue set for \( f \). If \( f \in L^1(\mathbb{T}) \) then the complement of the Lebesgue set for \( f \) is of Lebesgue measure zero. If \( f \) is continuous at \( x \), then \( x \) is a Lebesgue point for \( f \).

The space of all infinite sequences of complex numbers \( \langle c_n \rangle \), such that \( \sum_{n \in \mathbb{Z}} |c_n| < +\infty \), is denoted by \( l^1 \).

For \( a \in \mathbb{R} \), \( f_a \) is defined by \( f_a(t) = f(t + a) \), \( a^+ = \max(a, 0) \) and \( a^- = \max(-a, 0) \).

As usual \( \Re z \) and \( \Im z \) mean the real and imaginary parts of \( z \) respectively.

3. The main theorems.

Theorem 1. Let \( f \) be a continuous complex-valued function on \( \mathbb{T} \) with the property: there is \( \alpha \in \mathbb{R} \) such that \( \alpha \leq \arg \hat{f}(n) \leq \alpha + \pi/2 \) \((n \in \mathbb{Z})\). Then \( f \in A \). Also every \( f \in A \) is a linear combination of continuous functions on \( \mathbb{T} \) with the above property.

Proof. The second part of the theorem is obvious. To prove the first part, observe that we may assume \( \alpha = 0 \). For if \( \alpha \neq 0 \) then we consider the function \( g(x) = f(x)e^{-ia} \) for which \( 0 \leq \arg \hat{g}(n) \leq \pi/2 \). Next set
\[
F(x) = \frac{f(x) + \overline{f(-x)}}{2}, \quad G(x) = \frac{f(x) - \overline{f(-x)}}{2i}.
\]
Clearly both \( F \) and \( G \) are continuous. Also
\[
\hat{F}(n) = \Re \hat{f}(n) \geq 0, \quad \hat{G}(n) = \Im \hat{f}(n) \geq 0.
\]
It follows from the theorem stated in the introduction that \( F \in A \), \( G \in A \) so that \( f \in A \).

Theorem 2. Let \( f \in L^1(\mathbb{T}) \). Then \( f \in \mathcal{A} \) if and only if the following condition is satisfied:

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"there is a Lebesgue point \( \alpha \) for \( f \) such that both sequences
\[ \langle \Re \hat{f}_n(n) \rangle_{n \in \mathbb{Z}}, \quad \langle \Im \hat{f}_n(n) \rangle_{n \in \mathbb{Z} \} \text{ belong to } l^1."

Proof. Suppose \((*)\) is satisfied. We first consider the case \( \alpha = 0. \) For
\( N \) a positive integer, set
\[ \sigma_N(t) = \sum_{n=-N}^{N} (1 - |n|/N) \hat{f}(n)e^{int}. \]

By a theorem of Lebesgue we know that if \( t \) is a Lebesgue point for \( f \) then
\[ \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} f(n) = f(t). \]

Hence
\[ \lim_{N \to \infty} \sigma_N(0) = \lim_{N \to \infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n) = f(0) = \text{finite}. \]

Also
\[ \hat{f}(0) = \sum_{n=-N}^{N} (1 - |n|/N) \Re \hat{f}(n) + i \sum_{n=-N}^{N} (1 - |n|/N) \Im \hat{f}(n) \]
\[ = \sum_{n=-N}^{N} (1 - |n|/N) (\Re \hat{f}(n))^+ + i \sum_{n=-N}^{N} (1 - |n|/N) (\Im \hat{f}(n))^+ \]
\[ \quad - \sum_{n=-N}^{N} (1 - |n|/N) (\Re \hat{f}(n))^- - i \sum_{n=-N}^{N} (1 - |n|/N) (\Im \hat{f}(n))^- \]

If we let \( N \to \infty \) then the \( \sigma_N(0) \) are uniformly bounded because of (1) while
the last two sums of the right-hand side of (2) are bounded (more precisely
they converge) because of the hypothesis \((*)\). Therefore,
\[ \lim_{N \to \infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \Re \hat{f}(n) < +\infty, \quad \lim_{N \to \infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \Im \hat{f}(n) < +\infty. \]

Since the Cesàro summability of a series with nonnegative terms implies the
convergence of the series, it follows that \( \sum_{n \in \mathbb{Z}} \hat{f}(n) < +\infty: \) i.e., \( f \in \overset{\circ}{C} \).

Next assume \( \alpha \neq 0. \) Then 0 is a Lebesgue point for \( f_{\alpha} \) so that by
the result that we just proved we have \( \sum_{n \in \mathbb{Z}} |\hat{f}_\alpha(n)| < +\infty. \) Since \( \hat{f}_\alpha(n) = e^{i\alpha n}\hat{f}(n) \)
we have \( f \in \overset{\circ}{C}. \) Now suppose \( f \in \overset{\circ}{C} \) and let \( \alpha \) be any Lebesgue point for \( f. \)
Then
\[ \sum_{n \in \mathbb{Z}} |\hat{f}(n)| = \sum_{n \in \mathbb{Z}} |e^{i\alpha n}\hat{f}(n)| = \sum_{n \in \mathbb{Z}} |\hat{f}_\alpha(n)| < +\infty \]
and condition \((*)\) is clearly satisfied. \( \Box \)

Corollary 3. Let \( f \in L^1(T) \). Then \( f \) is equal almost everywhere to a
linear combination of positive definite functions if and only if condition (*) is satisfied.

Proof. This follows from Kahane's characterization of $A$ (see §1) and Herglotz's characterization of continuous functions with nonnegative coefficients as positive definite [2, p. 19].

Theorem 4. Let $f \in L^1(T)$. Then $f \in \mathfrak{A}$ if and only if the following condition is satisfied

"$f$ is essentially bounded in a neighborhood of some real number $a$, and both sequences $\langle (R_n f(n))^\sim \rangle_{n \in \mathbb{Z}}$, $\langle \langle f_m n(n) \rangle \rangle_{n \in \mathbb{Z}}$ belong to $l^1$".

Proof. Suppose that condition (**) is satisfied. We first consider the case $a = 0$. Using the notation of Theorem 2 we have:

$$\sigma_N(t) = \frac{1}{2\pi} \int_T f(y) K_N(t - y) dy$$

where

$$K_N(y) = \sum_{n = -N}^{N} \left( 1 - \frac{|n|}{N} \right) e^{iny} = \frac{\sin^2(N/2)y}{N \sin^2(y/2)}.$$

Next assume $|f(y)| \leq M$ a.e. for $y \in (-h, h)$ ($h > 0$). We have

$$\sigma_N(0) = \frac{1}{2\pi} \int_{-h}^{h} f(y) K_N(y) dy + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \right) + \frac{1}{2\pi} \int_{-h}^{h} \left( \right).$$

Observe that the first of the last three integrals is bounded by $M$, and that the other two converge to zero as $N \to \infty$ by the Lebesgue dominated convergence theorem. Therefore the $\sigma_N(0)$ are uniformly bounded. From this point on we proceed exactly as in the proof of Theorem 2, by letting $N \to \infty$ in (2). Thus $f \in \mathfrak{A}$. Now suppose $a \neq 0$. Then $f_a$ is essentially bounded in a neighborhood of the origin so that by the previous result, $f_a \in \mathfrak{A}$. Since $f(n) = e^{-ina} f_a(n)$ we have $f \in \mathfrak{A}$. Next assume $f \in \mathfrak{A}$. Then $f$ is essentially bounded on $T$ and condition (**) is clearly satisfied.

Corollary 5. Let $f \in L^1(T)$. Then $f$ is equal almost everywhere to a linear combination of positive definite functions if and only if (**) is satisfied. □

Remark. Call a numerical series $\Sigma(a_n + ib_n)$ "one-sidedly absolutely..."
convergent' if (at least one of $\Sigma a_n^+$ or $\Sigma a_n^-$) and (at least one of $\Sigma b_n^+$ or $\Sigma b_n^-$) is finite. The main theorems (2 and 4) are interesting because it is possible for $\Sigma e^{i\lambda} c_n$ to be one-sidedly absolutely convergent even when $\Sigma c_n$ is not. For example, $c_{2n} = 1 + i$, $c_{2n+1} = 1 - i$, $n = 0, 1, 2, \cdots$, and $\lambda = \pi/4$.

REFERENCES


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