AN EVERYWHERE DIVERGENT FOURIER-WALSH
SERIES OF THE CLASS \( L(\log^+ \log^+ L)^{1-\epsilon} \)

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ABSTRACT. Let \( \Phi \) be a function satisfying (a) \( \Phi(t) \geq 0 \), convex and increasing; (b) \( \Phi(t^{1/2}) \) is a concave function of \( t \), \( 0 \leq t < \infty \); and (c) \( \Phi(t) = o(t \log \log t) \) as \( t \to \infty \). We construct a function in the class
\[
\Phi(L) = \{ f \in L(0, 1): \int_0^1 \Phi(|f(x)|) dx < \infty \}
\]
whose Fourier-Walsh series diverges everywhere.

It is known that there exists a function in the class \( L(\log^+ \log^+ L)^{1-\epsilon} \) for \( \epsilon > 0 \) whose trigonometric series diverges almost everywhere [1]. Let \( \Phi \) be a function satisfying
(a) \( \Phi(t) \geq 0 \), convex and increasing in \( 0 < t < \infty \),
(b) \( \Phi(t^{1/2}) \) is a concave function of \( t \), \( 0 < t < \infty \), and
(c) \( \Phi(t) = o(t \log \log t) \) as \( t \to \infty \).

For the Walsh system, we will construct a function in the class
\[
\Phi(L) = \{ f \in L(0, 1): \int_0^1 \Phi(|f(x)|) dx < \infty \}
\]
whose Fourier-Walsh series diverges everywhere by refining Stein's construction [3] of a function in \( L(0, 1) \) with almost everywhere divergent Fourier-Walsh series.

We recall the definition of the Walsh system in the Paley enumeration. The Rademacher functions \( r_n(x) \) are defined by
\[
r_0(x) = 1 \quad (0 < x < \frac{1}{2}), \quad r_0(x) = -1 \quad (\frac{1}{2} \leq x < 1),
\]
(1)
\[
r_0(x + 1) = r_0(x), \quad r_n(x) = r_0(2^n x) \quad (n = 1, 2, \ldots).
\]

For each positive integer \( n \), there is a unique representation of the form
\[
n = \sum_{j=0}^{\infty} \epsilon_j 2^j, \quad \text{where } \epsilon_j = 0 \text{ or } 1.
\]
The Walsh functions in the Paley enumeration are then given by
\[ w_0(x) = 1, \quad w_n(x) = \prod_{j=0}^{\infty} [r_j(x)]^j. \]

Let \( x \) be any real number in \((0, 1)\). Then we have a unique representation of the form \( x = \sum_{n=1}^{\infty} x_n 2^{-n} \) with infinitely many \( x_n \neq 0 \), where \( x_n = 0 \) or 1.

We define
\[
(3) \quad x \dagger y = \sum_{n=1}^{\infty} |x_n - y_n| 2^{-n}
\]
where \( x = \sum_{n=1}^{\infty} x_n 2^{-n}, y = \sum_{n=1}^{\infty} y_n 2^{-n} \), \( x_n, y_n = 0 \) or 1 and the operation \( \dagger \) is called dyadic addition (see Fine [2]).

We denote the Dirichlet kernel and the partial sum of Fourier series of \( f(x) \) with respect to the Walsh functions in the Paley enumeration by
\[
(4) \quad \begin{align*}
D_n(x) &= \sum_{j=0}^{n-1} w_j(x), \\
S_n f(x) &= \sum_{j=0}^{n-1} C_j(f) w_j(x) = \int_0^1 f(t) D_n(x + t) dt,
\end{align*}
\]
where \( C_j = C_j(f) = \int_0^1 f(t) w_j(t) dt \) is the \( j \)th Fourier coefficient of \( f \).

The Lebesgue constant \( L_n \) is given by
\[
(5) \quad L_n = \int_0^1 |D_n(t)| dt.
\]
It is well known (see Fine [2]) that
\[
(6) \quad \lim_{n \to \infty} \sup \left( \frac{L_n}{\log n} \right) \geq \alpha > 0.
\]

An interval \( I \) with the length \( 2^{-n} \) is called a dyadic interval if the \((n-1)\)th Rademacher function \( r_{n-1}(t) \) is constant on \( I \).

First of all we want to prove the following lemma, from which our main theorem follows. A part of the proof of this lemma will use a technique of E. M. Stein in [3].

Lemma. For any fixed positive integer \( n \), there exists a set \( E_n \) such that
\[
\begin{align*}
(\text{i}) \quad & m(E_n) = 2^{-2N}, \quad \text{where} \quad 2^{N-1} \leq n < 2^N, \\
(\text{ii}) \quad & C_k(\mathcal{X}E_n(t)) = \int_0^1 \mathcal{X}E_n(t) w_k(t) dt = 0 \quad \text{if} \quad 0 < k < 2^N \quad \text{or} \quad k \geq 2^{N+2N}, \\
(\text{iii}) \quad & M\mathcal{X}E_n(x) = \sup_{n > 1} |S_n \mathcal{X}E_n(x)| \geq \frac{1}{2} L_n m(E_n),
\end{align*}
\]
where \( m(A) \) and \( \mathcal{X}_A \) denote the Lebesgue measure and the characteristic function of the set \( A \) respectively.
Proof. Let \( I_j = [(j - 1)2^{-N}, j2^{-N}) \) \((j = 1, 2, \ldots, 2^N)\). We will choose dyadic intervals \( d_j \) such that \( d_j \subseteq I_j \) and \( m(d_j) = 2^-(N+2^N) \) for all \( j = 1, 2, \ldots, 2^N \), and put \( E_n = \bigcup_{j=1}^{2^N} d_j \).

Then we get

\[
m(E_n) = \sum_{j=1}^{2^N} m(d_j) = 2^{-2^N}.
\]

We note that for any \( k \) with \( 0 < k < 2^N \)

\[
m\{t \in E_n; w_k(t) = 1\} = m\{t \in E_n; w_k(t) = -1\}
\]

and for any \( k \geq 2^N + 2^N \)

\[
m\{t \in d_j; w_k(t) = 1\} = m\{t \in d_j; w_k(t) = -1\}
\]

for all \( j = 1, 2, \ldots, 2^N \). Hence, we obtain

\[
C_k(\chi_{E_n}) = 0 \quad \text{if} \quad 0 < k < 2^N \quad \text{or} \quad k \geq 2^N + 2^N.
\]

It remains to choose the dyadic intervals \( d_j \), so that (iii) is satisfied.

We note that for \( n < 2^N \), \( D_n(t) = \sum_{j=0}^{n-1} w_j(t) \) is constant on \( I_j \) for each \( i = 1, 2, \ldots, 2^N \), and hence \( D_n(x + t) \) is constant as \( x \) and \( t \) vary over \( I_i \) and \( I_j \) respectively. Let \( D_n(I_i + I_j) \) denote the value of \( D_n(x + t) \) for \( x \in I_i \) and \( t \in I_j \), and

\[
\sigma(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}
\]

Consider the \( 2^N \)-tuples \( R_k \) \((1 \leq k \leq 2^N)\) such that

\[
R_k = (\sigma(D_n(I_k + I_{k+1})), \sigma(D_n(I_k + I_{k+2})), \ldots, \sigma(D_n(I_k + I_{2^N}))).
\]

We now define the dyadic interval \( d_j \) by

\[
d_j = [(j - 1)2^{-N}, j2^{-N}) + \sum_{i=1}^{2^N} \epsilon_{ji} 2^{-i} \quad \text{and} \quad (j - 1)2^{-N} + \sum_{i=1}^{2^N} \epsilon_{ji} 2^{-i} + 2^{-(N+2^N)}
\]

where \( \epsilon_{ji} \) \((1 \leq i \leq 2^N, 1 \leq j \leq 2^N)\) is either 0 or 1 and

\[
(-1)^{\epsilon_{ji}} = \sigma(D_n(I_i + I_j)).
\]

Hence, for all \( t \in d_j \) \((1 \leq j \leq 2^N)\)

\[
w_{2^N+i-1}(t)D_n(I_i + I_j) = r_{N+i-1}(t)D_n(I_i + I_j) = (-1)^{\epsilon_{ji}}D_n(I_i + I_j) \geq 0
\]

for each \( i \) with \( 1 \leq i \leq 2^N \).

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Now we set \( E_n = \bigcup_{j=1}^{2N} d_j \) and it remains to show that \( \chi_{E_n} \) satisfies condition (iii).

For any fixed \( x \in [0, 1) \), there exists a unique \( k \) such that \( x \in I_k \), and we set
\[
N_{kx} = n + 2^{N+k-1} \quad (2^{N-1} < n < 2^N, n \text{ fixed}).
\]
We again note that
\[
d_n(x^+t) \geq 0
\]
for all \( t \in d_j \) and \( j = 1, 2, \ldots, 2^N \). Since \( d_n(x^+t) \) is constant on each \( I_i \) \((1 \leq i \leq 2^N)\) and \( m(E_n) = 2^{-2N} \), we obtain, by applying (15),
\[
|S_{N_{kx}} \chi_{E_n}(x) - S_{2^{N+k-1}} \chi_{E_n}(x)| = \left| \int_0^1 \chi_{E_n}(t) d_n(x^+t)dt \right|
\]
(16)
\[
= \left| \sum_{j=1}^{2^N} \int_{d_j} w_{2^{N+k-1}}(t) d_n(x^+t)dt \right| = \sum_{j=1}^{2^N} \int_{d_j} |D_n(x^+t)|dt
\]
\[
= 2^{-2N} \sum_{j=1}^{2^N} \int_{I_j} |D_n(x^+t)|dt = m(E_n) \int_0^1 |D_n(x^+t)|dt = m(E_n) \cdot L_n.
\]
Thus, (16) implies \( M_\chi_{E_n}(x) \geq \frac{1}{2} L_n m(E_n) \). The lemma is proved.

Now we are ready to prove the following theorem:

**Theorem.** Let \( \Phi \) be a function satisfying conditions (a), (b) and (c).

Then there exists a function \( f \in \Phi(L(0, 1)) \) such that \( S_n f(x) \) diverges everywhere.

**Proof.** If we note (6) and properties of the function \( \Phi \), we may choose a sequence \( \{n_j\} \) of positive integers satisfying the following conditions:

(a) there is a constant \( A > 0 \) such that \( L_n \geq A \log n \),

(b) \( N_{j+1} \geq N_j + 2^N \), and

\( \Phi(\alpha_j) \leq j^{-2} \alpha_j (\log \log \alpha_j) \),

where \( 2^{N_j} \leq n_j < 2^{N_j} \), \( \alpha_j = 1/(\log n_j) m(E_n) \), \( m(E_n) = 2^{-2N} \) and the sets \( E_n \) are the same as in the lemma. It is easy to see that the sequence \( \{\alpha_n\} \) is a lacunary sequence and there exists a constant \( C \) such that
\[
\sum_{j=1}^{\infty} \alpha_j \leq C \alpha_n.
\]

Let \( f \) be the measurable function defined by
(18) \[ f(x) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_{n_j}}(x). \]

From the properties of \( \Phi \) and (17) we get

(19) \[ \Phi \left( \sum_{j=1}^{\infty} \alpha_j \chi_{E_{n_j}}(x) \right) \leq C \sum_{j=1}^{\infty} \Phi(\alpha_j) \chi_{E_{n_j}}(x). \]

In fact, if \( x \) does not belong to \( \bigcup_{j=1}^{\infty} E_{n_j} \) or \( x \) belongs to infinitely many \( E_{n_j} \)'s then both sides of (19) are equal to 0 or \( \infty \) respectively, and if \( x \) belongs to finitely many \( E_{n_j} \)'s then

\[ \Phi \left( \sum_{j=1}^{\infty} \alpha_j \chi_{E_{n_j}}(x) \right) = \Phi \left( \sum_{j=1}^{k} \alpha_j \chi_{E_{n_j}}(x) \right) \leq \Phi \left( \sum_{j=1}^{k} \alpha_j \chi_{E_{n_k}}(x) \right) \]

\[ \leq \Phi(C \alpha_k) \chi_{E_{n_k}}(x) \leq C \sum_{j=1}^{\infty} \Phi(\alpha_j) \chi_{E_{n_j}}(x) \]

where \( k = \text{max}\{j; x \in E_{n_j}\} < \infty \). Hence, we have

(20) \[ \int_{0}^{1} \Phi(f(x))dx \leq C \sum_{j=1}^{\infty} \Phi(\alpha_j)m(E_{n_j}) \leq C \sum_{j=1}^{\infty} \frac{1}{j^2} \alpha_j (\log \log \alpha_j)m(E_{n_j}) < \infty. \]

This implies \( f \in \Phi(L) \).

Now it remains to show that \( S_n f(x) \) diverges everywhere. Let \( x \) be a fixed point in \([0, 1)\).

For each positive integer \( k \), (14) and (16) imply that there exists a positive integer \( n_{kx} \) such that

(21) \[ n_{kx} = n_k + 2^{N_{kx}} \quad \text{with} \quad N_k \leq n_{kx} < N_k + 2^{N_k}, \]

and

(22) \[ |S_{n_{kx}} \chi_{E_{n_k}}(x) - S_{2^{N_{kx}}} \chi_{E_{n_k}}(x)| = L_{n_k} m(E_{n_k}). \]

If \( j \neq k \), we obtain

(23) \[ S_{n_{kx}} \chi_{E_{n_j}}(x) - S_{2^{N_{kx}}} \chi_{E_{n_j}}(x) = \sum_{i=2^{N_{kx}}}^{n_{kx}-1} C_i(\chi_{E_{n_j}}) \omega_i(x) = 0 \]

since part (ii) of the lemma implies \( C_i(\chi_{E_{n_j}}) = 0 \) if \( 2^{N_k} \leq i < 2^{N_k+1} \). A combination of (22), (23), (18) and (a) gives

\[ |S_{n_{kx}} f(x) - S_{2^{N_{kx}}} f(x)| = \alpha_k |S_{n_{kx}} \chi_{E_{n_k}}(x) - S_{2^{N_{kx}}} \chi_{E_{n_k}}(x)| \]

\[ = \alpha_k L_{n_k} m(E_{n_k}) = L_{n_k} /\log n_k \geq A > 0. \]
We finally get
\[
\limsup_{m, n \to \infty} |S_m f(x) - S_n f(x)| \geq A > 0
\]
for all \( x \in [0, 1) \), that is, the Fourier-Walsh series of \( f \in \Phi(L) \) diverges everywhere.

**Remark.** A theorem in E. M. Stein [3, Theorem 3] implies that if for every \( f \in \Phi(L) \)
\[
m \left\{ x \in (0, 1): \limsup_{n \to \infty} |S_n f(x)| < \infty \right\} > 0
\]
then there exists an absolute constant \( A \) such that for any \( y > 0 \)
\[
m \left\{ x \in (0, 1): \sup_{n \geq 1} |S_n f(x)| > y \right\} \leq \int_0^1 \Phi \left( \frac{A}{y} |f(x)| \right) dx.
\]

We may apply this theorem to prove the existence of a function in the class \( \Phi(L) \) whose Fourier-Walsh series diverges almost everywhere.

In fact, let \( f(x) = \chi_{E_n}(x) \) and \( y_n = \frac{1}{2} L_n m(E_n) \), where the set \( E_n \) is defined in the lemma. Then part (iii) of the Lemma implies
\[
m \left\{ x \in (0, 1): M_{\chi_{E_n}}(x) > y_n \right\} = 1
\]
for all positive integers \( n \), but for \( \epsilon, 0 < \epsilon < 1 \),
\[
\int_0^1 \Phi \left( \frac{A_{\chi_{E_n}}(x)}{y_n} \right) dx \leq \epsilon < 1
\]
for all sufficiently large \( n \) where the constant \( A \) is as same as in the inequality (26).

Thus, our theorem for the almost everywhere divergence follows.

**REFERENCES**


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