PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, July 1975

SOME FIXED POINT THEOREMS FOR CONDENSING MULTIFUNCTIONS IN LOCALLY CONVEX SPACES

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ABSTRACT. Let G be a nonempty subset of a locally convex space E such that $\mathrm{cl}(G)$ is convex and quasi-complete, and $f: \mathrm{cl}(G) \to E$ a continuous condensing multifunction. In this paper, several fixed point theorems are established if f satisfies some conditions on the boundary of G. The results herein extend some theorems of Reich [9] and generalize some of the well-known fixed point theorems.

The classical Tychonoff's fixed point theorem [10] has been extended to multifunctions by Browder [3], Fan [4], Glicksberg [5], Himmelberg [7] and several others. In a recent paper [9], Reich has proved some interesting fixed point theorems for condensing multifunctions and has given extensions of some of the results contained in [3] and [4]. In this paper, we prove several fixed point theorems for condensing multifunctions in a locally convex space. The main result of this paper (Theorem 2) is motivated by Reich [9] and extends a result in [9]. The results herein also generalize some well-known results (see [2], [8]).

1. Let X and Y be topological spaces. A multifunction $f: X \to Y$ is a point to set function such that for each $x \in X$, $f(x)$ is a nonempty subset of Y. The multifunction $f: X \to Y$ is (a) upper semicontinuous (u. s. c.) iff for each closed subset B of Y, the set $f^{-1}(B) = \{x: f(x) \cap B \neq \emptyset\}$ is a closed subset of X, (b) lower semicontinuous (l.s.c.) iff for each open subset $U \subseteq Y$, the set $f^{-1}(U)$ is an open subset of X, (c) continuous iff it is both u.s.c. and l.s.c., (d) point-compact (closed, convex) iff for each $x \in X$, $f(x)$ is a compact (closed, convex) subset of Y.

It follows immediately from the above that a multifunction $f: X \to Y$ is u.s.c. iff for each $x \in X$ and open $U \subseteq Y$ with $f(x) \subseteq U$, there is an open $V \subseteq X$ such that $x \in V$ and $f(V) = \bigcup\{f(z): z \in V\} \subseteq U$. It is l.s.c. iff for each open set $U \subseteq Y$ and each $x \in X$ with $f(x) \cap U \neq \emptyset$, there exists an

Received by the editors February 16, 1974.
AMS (MOS) subject classifications (1970). Primary 47H10, 54H20.
Key words and phrases. Multifunction, condensing, fixed point, quasi-complete.

open set $V \subseteq X$ such that $x \in V$ and $f(z) \cap U \neq \emptyset$ for each $z \in V$.

Proposition 1. *Let $f: X \to Y$ be a u.s.c., point-compact multifunction and $\{x_\alpha: \alpha \in \Gamma\}$ a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in f(x_\alpha)$ for each $\alpha \in \Gamma$, then there is a $y_0 \in f(x_0)$ and a subnet $\{y_\beta\}$ of the net $\{y_\alpha: \alpha \in \Gamma\}$ such that $y_\beta \to y_0$.*

Proof. Suppose that no subnet of the net $\{y_\alpha: \alpha \in \Gamma\}$ converges to a point in $f(x_0)$. Then for each $y \in f(x_0)$, there is an open neighborhood $U(y)$ of y such that $y_\alpha \notin U(y)$ eventually. Since $f(x_0)$ is compact, there is a finite subset $\{y_i: i = 1, 2, \ldots, n\} \subseteq f(x_0)$ such that

$$f(x_0) \subseteq \bigcup \{U(y_i): i = 1, 2, \ldots, n\} = U \tag{1}$$

and $y_\alpha \notin U$ eventually. Now, f being u.s.c., it follows from (1) that there is open neighborhood V of x_0 such that $f(V) \subseteq U$. However, this implies that $f(x_\alpha) \subseteq U$ eventually and hence, $y_\alpha \in U$ eventually, a contradiction.

2. Throughout this section let E denote a locally convex separated topological vector space and $\mathfrak{U}$ a base of absolutely convex neighborhoods of the origin θ. A subset X of E is totally bounded iff for each $U \in \mathfrak{U}$, there exists a finite subset $B \subseteq X$ such that $X \subseteq B + U$. For any subset A of E, let

$$Q(A) = \{U \in \mathfrak{U}: A \subseteq B + U \text{ for some totally bounded subset } B \text{ of } E\}.$$

Let $S \subseteq E$. A multifunction $f: S \to E$ is condensing iff $Q(A) \subsetneq Q(f(A))$ for each bounded but not totally bounded subset $A \subseteq S$ (see Himmelberg, Porter, Van Vleck [6]).

For a subset $S \subseteq E$, let ∂S denote the boundary of S in E, $\mathrm{co}(S)$ the convex hull of S. A closed subset B of E is quasi-complete if its closed bounded subsets are complete. It is clear that if X is a totally bounded subset of a quasi-complete subset B then $\mathrm{cl}(X)$ is compact.

The following lemma is similar to Theorem 1 in [6] and the proof therein works verbatim with the hypothesis of this lemma.

Lemma 1. *Let X be a quasi-complete convex subset of E and $f: X \to X$ be an u.s.c., point-compact, point-convex, condensing multifunction. If $f(X)$ is bounded, then f has a fixed point in X, that is, there is an $x_0 \in X$ such that $x_0 \in f(x_0)$.*

Theorem 1. *Let X be a convex, quasi-complete subset of E, and $f: X \to E$ an u.s.c., point-compact, point-convex, condensing multifunction. If $f(X)$ is bounded and $f(x) \cap X \neq \emptyset$ for each $x \in X$, then f has a fixed point in X.*

Proof. Let $F(x) = f(x) \cap X$. Then F satisfies the conditions of Lemma 1, and hence there is an $x_0 \in X$ such that $x_0 \in F(x_0) \subseteq f(x_0)$.

The following consequences of Theorem 1 are immediate.

Corollary 1. *Let X be a convex, quasi-complete subset of E and f: $X \to E$ be an u.s.c., point-compact, point-convex, condensing multifunction. If there exists a bounded subset K of X such that $f(x) \cap K \neq \emptyset$ for all $x \in X$, then f has a fixed point in X.*

Corollary 2. *Let X be a convex, quasi-complete subset of E and K a totally bounded subset of X. If f: $X \to E$ is an u.s.c., point-closed, point-convex multifunction such that $f(x) \cap K \neq \emptyset$ for all $x \in X$, then f has a fixed point in X.*

Since for any subset $G \subseteq E$, $\partial(\mathrm{cl}(G)) \subseteq \partial(G)$, the following result extends a recent result of Reich [9, Theorem 3.5].

Theorem 2. *Let G be a nonempty open subset of E such that* $\mathrm{cl}(G)$ *is convex and quasi-complete. If f:* $\mathrm{cl}(G) \to E$ *is a continuous, point-compact, point-convex, condensing multifunction satisfying the conditions* (a) *$f(G)$ is bounded,* (b) *there is a $w \in G$ such that for all $y \in \partial(\mathrm{cl}(G))$ and $z \in f(y)$, $z - w = m(y - w)$ implies $m \not> 1$, then f has a fixed point in* $\mathrm{cl}(G)$.

Proof. Let p be the Minkowski functional of the set $\mathrm{cl}(G) - w$. Define a mapping q: $\mathrm{cl}(G) \to (0, 1]$ by

$$q(x) = [\max\{1, \min\{p(y - w): y \in f(x)\}\}]^{-1}. \tag{3}$$

Note that since $f(x)$ is compact and p is continuous, the minimum in (3) exists at some element of $f(x)$. Define a multifunction g: $\mathrm{cl}(G) \to E$ by

$$g(x) = q(x)f(x) + (1 - q(x))w. \tag{4}$$

Note that for each $x \in \mathrm{cl}(G)$, there is a $y \in f(x)$ such that $p(y - w) = \min\{p(z - w): z \in f(x)\}$ and hence $z = q(x)y + (1 - q(x))w \in g(x)$. Moreover, for this y, it follows from (3) that $p(z - w) = q(x)p(y - w) \leq 1$. Therefore $z \in \mathrm{cl}(G)$ and hence $\mathrm{cl}(G) \cap g(x) \neq \emptyset$ for any $x \in \mathrm{cl}(G)$. We show that g satisfies the conditions of Theorem 1. It is clear from (4) that $g(x)$ is compact and convex for each x. Further, since $g(\mathrm{cl}(G)) \subseteq \mathrm{co}(f(\mathrm{cl}(G)) \cup \{w\})$, it follows by hypothesis (a) that $g(\mathrm{cl}(G))$ is bounded. Now, for any bounded but not totally bounded subset A,

$$Q(g(A)) \supseteq Q(\mathrm{co}(f(A) \cup \{w\})) = Q(f(A)) \supsetneqq Q(A). \tag{5}$$

It follows from (5) that g is condensing. We show that g is u.s.c. Let B

be a closed subset of E and let $A = g^{-1}(B)$. We show that A is closed. Let $x_0 \in \mathrm{cl}(A)$ and let a net $\{x_\alpha : \alpha \in \Gamma\}$ in A be convergent to x_0. For each $\alpha \in \Gamma$, choose $y_\alpha \in f(x_\alpha)$ and a $y_0 \in f(x_0)$ such that $p(y_\alpha - w) = \min\{p(y-w) : y \in f(x_\alpha)\}$ and $p(y_0 - w) = \min\{p(y-w): y \in f(x_0)\}$. It follows by Proposition 1 that there is a subnet $\{y_\beta : \beta \in \Gamma_1\}$ of the net $\{y_\alpha : \alpha \in \Gamma\}$ converging to a $y \in f(x_0)$. Now, since p is continuous, for any $\epsilon > 0$, there is $\beta_1 \in \Gamma_1$ such that for all $\beta \geq \beta_1$

$$p(y_0 - w) \leq p(y - w) \leq p(y_\beta - w) + \epsilon. \tag{6}$$

Let $U = \{z \in E : p(z - w) < p(y_0 - w) + \epsilon\}$. Then U is open and $y_0 \in U \cap f(x_0)$. Now, f being l.s.c., there is a neighborhood V of x_0 such that $f(z) \cap U \neq \emptyset$ for each $z \in V$ and hence there is a $\beta_2 \in \Gamma_1$ such that $f(x_\beta) \cap U \neq \emptyset$ for all $\beta \geq \beta_2$. This implies that for $\beta \geq \beta_2$

$$p(y_\beta - w) < p(y_0 - w) + \epsilon. \tag{7}$$

It follows now from (6) and (7) that for all $\beta \geq \max\{\beta_1, \beta_2\}$,

$$p(y_0 - w) \leq p(y - w) \leq p(y_\beta - w) + \epsilon < p(y_0 - w) + 2\epsilon. \tag{8}$$

Since $\epsilon > 0$ is arbitrary, we have from (8) that $p(y_0 - w) = p(y - w)$ and $q(x_\beta) \to q(x_0)$. Since for any $\beta \in \Gamma_1$, $g(x_\beta) \cap B \neq \emptyset$, there is $z_\beta \in f(x_\beta)$ such that

$$q(x_\beta)z_\beta + (1 - q(x_\beta))w \in B. \tag{9}$$

By Proposition 1, there is a subnet $\{z_\delta\}$ of the net $\{z_\beta\}$ and a $z_0 \in f(x_0)$ such that $z_\delta \to z_0$. Since $q(x_\delta) \to q(x_0)$, therefore, from (9) it follows that $g(x_0) \cap B \neq \emptyset$, that is $x_0 \in A$. Thus, g is an u.s.c. multifunction, and satisfies the conditions of Theorem 1, and hence there is $u \in \mathrm{cl}(G)$ with $u \in g(u)$. We assert that $\alpha = \min\{p(y - w): y \in f(u)\} \leq 1$. Suppose $\alpha > 1$. Let $u = q(u)y_1 + (1 - q(u))w \in g(u)$ for some $y_1 \in f(u)$. By our assumption $p(y_1 - w) > 1$, and hence by the definition of q, it follows that $p(u - w) = q(u)\,p(y_1 - w) \geq 1$. Since $u \in \mathrm{cl}(G)$, the last inequality implies that $u \in \partial(\mathrm{cl}(G))$. Also $y_1 - w = [q(u)]^{-1}(u - w)$, $[q(u)]^{-1} > 1$. This contradicts hypothesis (b). Thus $\alpha \leq 1$ and hence $q(u) = 1$. This implies that $u \in g(u) = f(u)$. This establishes the proof of Theorem 2.

Corollary 3. *Let G be a nonempty open subset of E such that* $\mathrm{cl}(G)$ *is convex and quasi-complete. Suppose f:* $\mathrm{cl}(G) \to E$ *is a continuous, point-compact, point-convex, condensing multifunction such that* (i) *$f(G)$ is bounded, and* (ii) *$f(x) \subseteq \mathrm{cl}(G)$ for all $x \in \partial(\mathrm{cl}(G))$. Then f has a fixed point in* $\mathrm{cl}(G)$.

Theorem 3. *Let G be a nonempty open subset of E with* $\mathrm{cl}(G)$ *convex. Let* $f\colon \mathrm{cl}(G) \to E$ *be a continuous, point-closed, point-convex multifunction satisfying the conditions* (i) $\mathrm{cl}(f(G))$ *is compact, and* (ii) *there is a* $w \in G$ *such that for each* $y \in \partial(\mathrm{cl}(G))$ *and* $z \in f(y)$, $z - w = m(y - w)$ *implies* $m \not> 1$, *then f has a fixed point in* $\mathrm{cl}(G)$.

Proof. Define the functions q and g as in Theorem 2. Then g is u.s.c. and $g(x) \cap \mathrm{cl}(G) \neq \varnothing$ for any $x \in \mathrm{cl}(G)$. Since

$$g(\mathrm{cl}(G)) \subseteq [0, 1]\,\mathrm{cl}(f(G)) + [0, 1]w \tag{10}$$

and the right side of (10) is compact, therefore $\mathrm{cl}(g(\mathrm{cl}(G)))$ is compact. Define a multifunction $h\colon \mathrm{cl}(G) \to \mathrm{cl}(G)$ by $h(x) = g(x) \cap \mathrm{cl}(G)$. Then h is u.s.c. and $h(\mathrm{cl}(G)) \subseteq g(\mathrm{cl}(G))$. Thus, by Himmelberg's theorem [7, Theorem 2], there is a $u \in \mathrm{cl}(G)$ such that $u \in h(u)$. It can now be shown, as in Theorem 2, that $u \in h(u) \subseteq g(u) = f(u)$.

The following consequence of Theorem 3 is an extension of a recent result of Porter [8] and generalizes a result of Singball [2].

Corollary 4. *Let G be a nonempty open subset of E such that* $\mathrm{cl}(G)$ *is convex. If* $f\colon \mathrm{cl}(G) \to E$ *is a continuous, point-closed and point-convex multifunction such that* $\mathrm{cl}(f(G))$ *is compact and* $f(\partial(\mathrm{cl}(G))) \subseteq G$, *then f has a fixed point in* $\mathrm{cl}(G)$.

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