SOME FIXED POINT THEOREMS FOR CONDENSING MULTIFUNCTIONS IN LOCALLY CONVEX SPACES

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ABSTRACT. Let $G$ be a nonempty subset of a locally convex space $E$ such that $\text{cl}(G)$ is convex and quasi-complete, and $f: \text{cl}(G) \rightarrow E$ a continuous condensing multifunction. In this paper, several fixed point theorems are established if $f$ satisfies some conditions on the boundary of $G$. The results herein extend some theorems of Reich [9] and generalize some of the well-known fixed point theorems.

The classical Tychonoff's fixed point theorem [10] has been extended to multifunctions by Browder [3], Fan [4], Glicksberg [5], Himmelberg [7] and several others. In a recent paper [9], Reich has proved some interesting fixed point theorems for condensing multifunctions and has given extensions of some of the results contained in [3] and [4]. In this paper, we prove several fixed point theorems for condensing multifunctions in a locally convex space. The main result of this paper (Theorem 2) is motivated by Reich [9] and extends a result in [9]. The results herein also generalize some well-known results (see [2], [8]).

1. Let $X$ and $Y$ be topological spaces. A multifunction $f: X \rightarrow Y$ is a point to set function such that for each $x \in X$, $f(x)$ is a nonempty subset of $Y$. The multifunction $f: X \rightarrow Y$ is (a) upper semicontinuous (u. s. c.) iff for each closed subset $B$ of $Y$, the set $f^{-1}(B) = \{x : f(x) \cap B \neq \emptyset\}$ is a closed subset of $X$, (b) lower semicontinuous (l.s.c.) iff for each open subset $U \subseteq Y$, the set $f^{-1}(U)$ is an open subset of $X$, (c) continuous iff it is both u.s.c. and l.s.c., (d) point-compact (closed, convex) iff for each $x \in X$, $f(x)$ is a compact (closed, convex) subset of $Y$.

It follows immediately from the above that a multifunction $f: X \rightarrow Y$ is u.s.c. iff for each $x \in X$ and open $U \subseteq Y$ with $f(x) \subseteq U$, there is an open $V \subseteq X$ such that $x \in V$ and $f(V) = \bigcup f(z) : z \in V \subseteq U$. It is l.s.c. iff for each open set $U \subseteq Y$ and each $x \in X$ with $f(x) \cap U \neq \emptyset$, there exists an
open set $V \subseteq X$ such that $x \in V$ and $f(z) \cap U \neq \emptyset$ for each $z \in V$.

**Proposition 1.** Let $f: X \to Y$ be a u.s.c., point-compact multifunction and $\{x_\alpha: \alpha \in \Gamma\}$ a net in $X$ such that $x_\alpha \to x_0$. If $y_\alpha \in f(x_\alpha)$ for each $\alpha \in \Gamma$, then there is a $y_0 \in f(x_0)$ and a subnet $\{y_\beta\}$ of the net $\{y_\alpha: \alpha \in \Gamma\}$ such that $y_\beta \to y_0$.

**Proof.** Suppose that no subnet of the net $\{y_\alpha: \alpha \in \Gamma\}$ converges to a point in $f(x_0)$. Then for each $y \in f(x_0)$, there is an open neighborhood $U(y)$ of $y$ such that $y_\alpha \notin U(y)$ eventually. Since $f(x_0)$ is compact, there is a finite subset $\{y_i: i = 1, 2, \ldots, n\} \subseteq f(x_0)$ such that

$$f(x_0) \subseteq \bigcup U(y_i): i = 1, 2, \ldots, n = U$$

and $y_\alpha \notin U$ eventually. Now, $f$ being u.s.c., it follows from (1) that there is open neighborhood $V$ of $x_0$ such that $f(V) \subseteq U$. However, this implies that $f(x_\alpha) \subseteq U$ eventually and hence, $y_\alpha \in U$ eventually, a contradiction.

2. Throughout this section let $E$ denote a locally convex separated topological vector space and $\mathcal{U}$ a base of absolutely convex neighborhoods of the origin $\theta$. A subset $X$ of $E$ is totally bounded iff for each $U \in \mathcal{U}$, there exists a finite subset $B \subseteq X$ such that $X \subseteq B + U$. For any subset $A$ of $E$, let

$$Q(A) = \{U \in \mathcal{U}: A \subseteq B + U \text{ for some totally bounded subset } B \text{ of } E\}.$$ 

Let $S \subseteq E$. A multifunction $f: S \to E$ is condensing iff $Q(A) \subseteq Q(f(A))$ for each bounded but not totally bounded subset $A \subseteq S$ (see Himmelberg, Porter, Van Vleck [6]).

For a subset $S \subseteq E$, let $\partial S$ denote the boundary of $S$ in $E$, $\text{co}(S)$ the convex hull of $S$. A closed subset $B$ of $E$ is quasi-complete if its closed bounded subsets are complete. It is clear that if $X$ is a totally bounded subset of a quasi-complete subset $B$ then $\text{cl}(X)$ is compact.

The following lemma is similar to Theorem 1 in [6] and the proof therein works verbatim with the hypothesis of this lemma.

**Lemma 1.** Let $X$ be a quasi-complete convex subset of $E$ and $f: X \to X$ be an u.s.c., point-compact, point-convex, condensing multifunction. If $f(X)$ is bounded, then $f$ has a fixed point in $X$, that is, there is an $x_0 \in X$ such that $x_0 \in f(x_0)$.

**Theorem 1.** Let $X$ be a convex, quasi-complete subset of $E$, and $f: X \to E$ an u.s.c., point-compact, point-convex, condensing multifunction. If $f(X)$ is bounded and $f(x) \cap X \neq \emptyset$ for each $x \in X$, then $f$ has a fixed point in $X$. 

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Proof. Let \( F(x) = f(x) \cap X \). Then \( F \) satisfies the conditions of Lemma 1, and hence there is an \( x_0 \in X \) such that \( x_0 \in F(x_0) \subseteq f(x_0) \).

The following consequences of Theorem 1 are immediate.

Corollary 1. Let \( X \) be a convex, quasi-complete subset of \( E \) and \( f: X \to E \) be an u.s.c., point-compact, point-convex, condensing multifunction. If there exists a bounded subset \( K \) of \( X \) such that \( f(x) \cap K \neq \emptyset \) for all \( x \in X \), then \( f \) has a fixed point in \( X \).

Corollary 2. Let \( X \) be a convex, quasi-complete subset of \( E \) and \( K \) a totally bounded subset of \( X \). If \( f: X \to E \) is an u.s.c., point-closed, point-convex multifunction such that \( f(x) \cap K \neq \emptyset \) for all \( x \in X \), then \( f \) has a fixed point in \( X \).

Since for any subset \( G \subseteq E \), \( \partial(\text{cl}(G)) \subseteq \partial(G) \), the following result extends a recent result of Reich [9, Theorem 3.5].

Theorem 2. Let \( G \) be a nonempty open subset of \( E \) such that \( \text{cl}(G) \) is convex and quasi-complete. If \( f: \text{cl}(G) \to E \) is a continuous, point-compact, point-convex, condensing multifunction satisfying the conditions (a) \( f(G) \) is bounded, (b) there is a \( w \in G \) such that for all \( y \in \partial(\text{cl}(G)) \) and \( z \in f(y) \), \( z - w = m(y - w) \) implies \( m \neq 1 \), then \( f \) has a fixed point in \( \text{cl}(G) \).

Proof. Let \( p \) be the Minkowski functional of the set \( \text{cl}(G) - w \). Define a mapping \( q: \text{cl}(G) \to [0, 1] \) by

\[
q(x) = \left[ \max \{1, \min \{p(y - w) : y \in f(x)\}\} \right]^{-1}.
\]

Note that since \( f(x) \) is compact and \( p \) is continuous, the minimum in (3) exists at some element of \( f(x) \). Define a multifunction \( g: \text{cl}(G) \to E \) by

\[
g(x) = q(x)f(x) + (1 - q(x))w.
\]

Note that for each \( x \in \text{cl}(G) \), there is a \( y \in f(x) \) such that \( p(y - w) = \min \left\{ p(z - w) : z \in f(x) \right\} \) and hence \( z = q(x)y + (1 - q(x))w \in g(x) \). Moreover, for this \( y \), it follows from (3) that \( p(z - w) = q(x)p(y - w) \leq 1 \). Therefore \( z \in \text{cl}(G) \) and hence \( \text{cl}(G) \cap g(x) \neq \emptyset \) for any \( x \in \text{cl}(G) \). We show that \( g \) satisfies the conditions of Theorem 1. It is clear from (4) that \( g(x) \) is compact and convex for each \( x \). Further, since \( g(\text{cl}(G)) \subseteq \text{co}(f(\text{cl}(G)) \cup \{w\}) \), it follows by hypothesis (a) that \( g(\text{cl}(G)) \) is bounded. Now, for any bounded but not totally bounded subset \( A \),

\[
Q(g(A)) \supset Q(\text{co}(f(A) \cup \{w\})) = Q(f(A)) \supset Q(A).
\]

It follows from (5) that \( g \) is condensing. We show that \( g \) is u.s.c. Let \( B \)
be a closed subset of $E$ and let $A = g^{-1}(B)$. We show that $A$ is closed.

Let $x_0 \in \text{cl}(A)$ and let a net $\{x_\alpha : \alpha \in \Gamma\}$ in $A$ be convergent to $x_0$. For each $\alpha \in \Gamma$, choose $y_\alpha \in f(x_\alpha)$ and $y_0 \in f(x_0)$ such that $p(y_\alpha - w) = \min\{p(y - w) : y \in f(x_\alpha)\}$ and $p(y_0 - w) = \min\{p(y - w) : y \in f(x_0)\}$. It follows by Proposition 1 that there is a subnet $\{y_\beta : \beta \in \Gamma_1\}$ of the net $\{y_\alpha : \alpha \in \Gamma\}$ converging to a $y \in f(x_0)$. Now, since $p$ is continuous, for any $\varepsilon > 0$, there is $\beta_1 \in \Gamma_1$ such that for all $\beta \geq \beta_1$

$$p(y_\beta - w) < p(y_0 - w) + \varepsilon.$$  

Let $U = \{z \in E : p(z - w) < p(y_0 - w) + \varepsilon\}$. Then $U$ is open and $y_0 \in U \cap f(x_0)$. Now, $f$ being l.s.c., there is a neighborhood $V$ of $x_0$ such that $f(z) \cap U \neq \emptyset$ for each $z \in V$ and hence there is a $\beta_2 \in \Gamma_1$ such that $f(x_\beta) \cap U \neq \emptyset$ for all $\beta \geq \beta_2$. This implies that for $\beta \geq \beta_2$

$$p(y_\beta - w) < p(y_0 - w) + \varepsilon.$$  

It follows now from (6) and (7) that for all $\beta \geq \max\{\beta_1, \beta_2\}$

$$p(y_{\beta} - w) \leq p(y - w) \leq p(y_{\beta} - w) + \varepsilon < p(y_0 - w) + 2\varepsilon.$$  

Since $\varepsilon > 0$ is arbitrary, we have from (8) that $p(y_\beta - w) = p(y - w)$ and $q(x_\beta) \to q(x_0)$. Since for any $\beta \in \Gamma_1$, $g(x_\beta) \cap B \neq \emptyset$, there is $z_\beta \in f(x_\beta)$ such that

$$q(x_\beta)z_\beta + (1 - q(x_\beta))w \in B.$$  

By Proposition 1, there is a subnet $\{z_{\gamma}\}$ of the net $\{z_\beta\}$ and a $z_0 \in f(x_0)$ such that $z_{\gamma} \to z_0$. Since $q(x_0) \to q(x_0)$, therefore, from (9) it follows that $g(x_0) \cap B \neq \emptyset$, that is $x_0 \in A$. Thus, $g$ is an u.s.c. multifunction, and satisfies the conditions of Theorem 1, and hence there is $u \in \text{cl}(G)$ with $u \in g(u)$. We assert that $\alpha = \min\{p(y - w) : y \in f(u)\} \leq 1$. Suppose $\alpha > 1$. Let $u = q(u)y_1 + (1 - q(u))w \in g(u)$ for some $y_1 \in f(u)$. By our assumption $p(y_1 - w) > 1$, and hence by the definition of $q$, it follows that $p(u - w) = q(u)p(y_1 - w) \geq 1$. Since $u \in \text{cl}(G)$, the last inequality implies that $u \in \partial(\text{cl}(G))$. Also $y_1 - w = [q(u)]^{-1}(u - w)$, so $[q(u)]^{-1} > 1$. This contradicts hypothesis (b). Thus $\alpha \leq 1$ and hence $q(u) = 1$. This implies that $u \in g(u) = f(u)$. This establishes the proof of Theorem 2.

**Corollary 3.** Let $G$ be a nonempty open subset of $E$ such that $\text{cl}(G)$ is convex and quasi-complete. Suppose $f : \text{cl}(G) \to E$ is a continuous, point-compact, point-convex, condensing multifunction such that (i) $f(G)$ is bounded, and (ii) $f(x) \subseteq \text{cl}(G)$ for all $x \in \partial(\text{cl}(G))$. Then $f$ has a fixed point in $\text{cl}(G)$.
Theorem 3. Let $G$ be a nonempty open subset of $E$ with $\text{cl}(G)$ convex. Let $f: \text{cl}(G) \to E$ be a continuous, point-closed, point-convex multifunction satisfying the conditions (i) $\text{cl}(f(G))$ is compact, and (ii) there is a $w \in G$ such that for each $y \in \partial(\text{cl}(G))$ and $z \in f(y)$, $z - w = m(y - w)$ implies $m \neq 1$, then $f$ has a fixed point in $\text{cl}(G)$.

Proof. Define the functions $q$ and $g$ as in Theorem 2. Then $g$ is u.s.c. and $g(x) \cap \text{cl}(G) \neq \emptyset$ for any $x \in \text{cl}(G)$. Since

$$g(\text{cl}(G)) \subseteq [0, 1] \text{cl}(f(G)) + [0, 1]w$$

and the right side of (10) is compact, therefore $\text{cl}(g(\text{cl}(G)))$ is compact. Define a multifunction $h: \text{cl}(G) \to \text{cl}(G)$ by $h(x) = g(x) \cap \text{cl}(G)$. Then $h$ is u.s.c. and $h(\text{cl}(G)) \subseteq g(\text{cl}(G))$. Thus, by Himmelberg's theorem [7, Theorem 2], there is a $u \in \text{cl}(G)$ such that $u \in h(u)$. It can now be shown, as in Theorem 2, that $u \in h(u) \subseteq g(u) = f(u)$.

The following consequence of Theorem 3 is an extension of a recent result of Porter [8] and generalizes a result of Singball [2].

Corollary 4. Let $G$ be a nonempty open subset of $E$ such that $\text{cl}(G)$ is convex. If $f: \text{cl}(G) \to E$ is a continuous, point-closed and point-convex multifunction such that $\text{cl}(f(G))$ is compact and $f(\partial(\text{cl}(G))) \subseteq G$, then $f$ has a fixed point in $\text{cl}(G)$.

REFERENCES

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