A RANK THEOREM FOR INFINITE DIMENSIONAL SPACES

J. P. HOLMES

Abstract. Suppose $X$ is a Banach space, $U$ is an open set of $X$ containing 0, and $f$ is a continuously differentiable function from $U$ into $X$ satisfying $f(0) = 0$ and $f'(0)^2 = f'(0)$. An additional hypothesis is given for $f$ which, in case $X$ is finite dimensional, is equivalent to assuming $\operatorname{rank} f'(x) = \operatorname{rank} f'(0)$ for all $x$ in some neighborhood of 0. Under this hypothesis one obtains a local factorization of $f$ into $h_1 \circ f'(0) \circ h_2$ where each of $h_1$ and $h_2$ is a continuously differentiable homeomorphism. In addition there is a neighborhood of 0 in $f^{-1}(\{0\})$ which is the image of a continuously differentiable retraction. An application of these results to the theory of differentiable multiplications is given.

Condition I. There is a neighborhood $W$ of 0 in $X$ so that $(f'(0)|f(W))$ is one to one.

Condition II. There is a neighborhood $W$ of 0 in $X$ so that if $V$ is a neighborhood of 0 in $f'(0)(X)$ then $f(V)$ is a neighborhood of 0 in $f(W)$.

Theorem 1. $f$ satisfies Condition I if and only if $f$ satisfies Condition II.
Proof. Let $N$ denote the image of $f'(0)$. $N$ is closed since $f'(0)^2 = f'(0)$ and hence is a Banach space. $(f'(0) \circ f|N)'(0) = (f'(0) \circ f'(0)|N) = (f'(0)|N)$ is the identity function on $N$. Thus, by the inverse function theorem [1, p. 268], there are neighborhoods $A$ and $B$ of 0 in $N$ so that $(f'(0) \circ f|A)$ is a homeomorphism onto $B$ and $(f'(0) \circ f|A)^{-1}$ is continuously differentiable on $B$.

Now suppose $f$ satisfies Condition I and $W'$ is a neighborhood of 0 in $X$ so that $(f'(0)|f(W'))$ is one to one. Choose a neighborhood $W$ of 0 in $X$ so that $W$ is contained in $W'$ and $W \cap N$ is contained in $A$.

Suppose $V$ is a neighborhood of 0 in $N$ and $V$ is contained in $W$. $f'(0)(f(V))$ is a neighborhood of 0 in $N$ since $V$ is contained in $A$. Let $C = f'(0)^{-1}(f'(0)(f(V))) \cap f(W)$, and suppose $y$ is in $C$. $f'(0)(y)$ is in $B$ so there is an $x$ in $A$ so that $f'(0)(y) = f'(0)(f(x))$. But each of $f(x)$ and $y$ is in $f(W)$ so $f(x) = y$ since $f(W)$ is contained in $f(W')$.

Suppose $f$ satisfies Condition II and choose $W'$ so that if $V$ is contained in $W'$ and $V$ is a neighborhood of 0 in $N$ then $f(V)$ is a neighborhood of 0 in $f(W')$. Since $f(A \cap W')$ is a neighborhood of 0 in $f(W')$ we may choose the neighborhood $W$ of 0 in $X$ so that $f(W)$ is contained in $f(A \cap W')$.

If each of $x$ and $y$ is in $f(W)$ there are members $z$ and $w$ of $A$ so that $f(z) = x$ and $f(w) = y$. If $f'(0)(x) = f'(0)(y)$ then $f'(0)(f(z)) = f'(0)(f(w))$ and hence $z = w$ since each of $z$ and $w$ is in $A$. Thus $(f'(0)|f(W))$ is one to one.

Note. If $A$ is chosen as before and $C$ is a closed subset of $A$ then $(f|C)$ is a homeomorphism onto $(f|(C))$. This is a consequence of the following observations. If $\{a_n\}$ is a sequence in $C$ and $\{(f(a_n))\}$ converges then $\{(f'(0)(a_n))\}$ converges to some member $y$ of $f'(0)(f(C))$. Thus $\{a_n\}$ converges to $(f'(0)\circ f|A)^{-1}(y)$ and $(f|C)^{-1}$ is continuous.

**Theorem 2.** If $f(f(x)) = f(x)$ for each $x$ in $U \cap f^{-1}(U)$ then $f$ satisfies Condition II.

This is a consequence of Lemma 3 in [2].

**Theorem 3.** If $f$ satisfies Condition I then there is a neighborhood $C$ of 0 in $X$ and continuously differentiable homeomorphisms $h_1$ and $h_2$, each from a neighborhood of 0 in $X$ onto a neighborhood of 0 in $X$, so that $(f|C) = (h_1 \circ f'(0) \circ h_2|C)$.

**Proof.** Since $f$ satisfies Conditions I and II there is a neighborhood $W$ of 0 in $X$ so that if $V$ is a neighborhood of 0 in $N$ and $V$ is contained in $W$ then $f(V)$ is a neighborhood of 0 in $f(W)$ and $(f'(0)|f(W))$ is one to one.
Choose neighborhoods $A$ and $B$ of 0 in $N$ so that $A$ is contained in $W$, $(f'(0) \circ f|A)$ is a homeomorphism onto $B$, and $(f'(0) \circ f|A)^{-1}$ is continuously differentiable on $B$.

Define $K_2$ on $W$ by $K_2(x) = f(x) + (I - f'(0))(x)$. (I denotes the identity function on $X$.) $K_2'(0) = I$ so by the inverse function theorem, there are neighborhoods $D$ and $E$ of 0 in $X$ so that $(K_2|D) = h_2$ is a homeomorphism onto $E$ and $f'(0)(E)$ is contained in $B$.

Define $K_1$ by
\[
K_1(x) = (f'(0)f(A))^{-1}(f'(0)(x)) + (I - f'(0))(x)
\]
for each $x$ in $f'(0)^{-1}(A \cap B)$. $K_1$ is well defined since $f(A)$ is contained in $f(W)$. If $x$ is in $A \cap B$ then
\[
(f|A) \circ (f'(0) \circ f|A)^{-1}(x) = (f'(0)f(A))^{-1}(x).
\]
Thus $(f'(0)f(A))^{-1} \circ f'(0)$ is continuously differentiable on $\text{dom}(K_1)$.

In particular, $K_1'(0) = I$, so by the inverse function theorem there are neighborhoods $F$ and $G$ of 0 in $X$ so that $h_1 = (K_1|F)$ is a homeomorphism onto $G$.

Choose $C$, a neighborhood of 0 in $X$, so that $C$ is contained in $D$, $f'(0)(h_2(C))$ is contained in $\text{dom}(h_2)$, and $(C)$ is contained in $f(A)$. (This last condition can be arranged since $f(A)$ is a neighborhood of 0 in $f(W)$.)

If $x$ is in $C$ then
\[
h_1(f'(0)(h_2(x))) = (f'(0)f(A))^{-1}(f'(0)(f(x))) = f(x).
\]
Compare this conclusion with that of the rank theorem [1, p. 277].

Suppose $X$ is finite dimensional, $V$ is an open set of $X$, $x_0$ is in $V$, and $g$ is a continuously differentiable function from $V$ to $X$ satisfying $\text{rank}(g'(x)) = \text{rank}(g'(x_0))$ for each $x$ in $V$. There is a linear homeomorphism $T$ from $X$ into $X$ so that the function $f$ defined by $f(x) = T(g(x + x_0) - g(x_0))$ for each $x$ in $U = V - x_0$ satisfies $f'(0)^2 = f'(0)$, $f(0) = 0$, and $\text{rank}(f'(x)) = \text{rank}(f'(0))$ for each $x$ in $U$. Thus, the following theorem may be applied to $g$.

**Theorem 4.** If $X$ is finite dimensional then $f$ satisfies Condition 1 if and only if $\text{rank}(f'(x)) = \text{rank}(f'(0))$ for each $x$ in some neighborhood of $0$.

**Proof.** Suppose $X$ is finite dimensional and $f$ satisfies Condition 1. Choose homeomorphisms $h_1$ and $h_2$ as in the conclusion of Theorem 3.

Since $h_1'(0) = h_2'(0) = I$ there is a neighborhood $D$ of 0 in $X$ so that if $x$ is in $D$ then each of $h_1'(f'(0)(h_2(x)))$ and $h_2'(x)$ is invertible. If $x$ is in $D$ then $f'(x) = h_1'(f'(0)(h_2(x))) \circ f'(0) \circ h_2'(x)$, and hence $\text{rank}(f'(x)) = \text{rank}(f'(0))$. 

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Now suppose \( X \) is finite dimensional, \( D \) is a neighborhood of 0 in \( X \), and \( \text{rank}(f'(x)) = \text{rank}(f'(0)) \) for each \( x \) in \( D \). Let \( N = f'(0)(X) \) and \( M = \ker(f'(0)) = (I - f'(0))(X) \). Define \( h \) from \( D \) into \( N \times M \) by \( h(x) = (f'(0)f(x), (I - f'(0))(x)) \). Regard \( N \times M \) as a Banach space in the usual way and note that \( h'(0) = f'(0) \times (I - f'(0)) \) is a linear homeomorphism onto \( N \times M \). By the inverse function theorem, we may choose \( E \) a neighborhood of 0 in \( X \) so that \( (b|E) \) is a homeomorphism onto a neighborhood \( (0, 0) \) in \( N \times M \), \( (b|E)^{-1} \) is continuously differentiable on \( b(E) \), and \( [(b|E)^{-1}]'(x) = [h'(0)]^{-1}[(b|E)^{-1}(x))]^{-1} \) for each \( x \) in \( h(E) \).

If \( x \) is in \( E \) then \( h'(x)(n) = (f'(0) \circ f'(x)(n), 0) \) for each \( n \) in \( N \). \( h'(x) \) is invertible so \( \text{dim}(h'(x)(N)) = \text{dim}(N) = \text{dim}(N \times \{0\}) \). Thus \( f'(0) \circ f'(x)(N) = N \). This implies that each of \( (f'(0)f'(x)(X)) \) and \( (f'(x)|N) \) is one to one onto its image since \( \text{rank}(f'(x)) = \text{rank}(f'(0)) \).

If \( x \) is in \( E \) then \( f(x) = f(h^{-1}(h(x))) \). Hence, by the chain rule,
\[
(f'(x)) = (f \circ h^{-1})'(h(x)) \circ h'(x)
\]
\[
= (D_1 f \circ h^{-1})(h(x)) \circ f'(0) \circ f'(x) + (D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0)).
\]

So
\[
(D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0)) = f'(x) - (D_1 f \circ h^{-1})(h(x)) \circ f'(0) \circ f'(x)
\]
\[
= [(f'(0)f'(x)(X))^{-1} - (D_1 f \circ h^{-1})(h(x))]/f'(0) \circ f'(x).
\]

If \( n \) is in \( N \) then there is a \( y \) in \( N \) so that \( f'(0)(f(x)(y)) = n \). Hence
\[
[(f'(0)|\text{im}(f'(x)))^{-1} - (D_1 f \circ h^{-1})(h(x))](n)
\]
\[
= (D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0))(y) = 0.
\]
Thus \( (D_2 f \circ h^{-1})(h(x)) \circ (I - f'(0))(y) = 0 \) for each \( y \) in \( X \) and \( (D_2 f \circ h^{-1})(h(x)) = 0 \) for each \( x \) in \( E \).

Choose neighborhoods \( F \) and \( G \) of 0 in \( N \) and \( M \), respectively, so that \( G \) is convex and \( F \times G \) is contained in \( b(E) \). If each of \( (x, y) \) and \( (x, z) \) is in \( F \times G \) then
\[
f(h^{-1}(x, y)) - f(h^{-1}(x, z)) = \int_0^1 dt[(f' \circ h^{-1})'(x, z + t(y - z))(0, y - z)]
\]
\[
= \int_0^1 dt[(D_1 f \circ h^{-1})(x, z + t(y - z))(0)
\]
\[
+ (D_2 f \circ h^{-1})(x, z + t(y - z))(y - z)] = 0
\]
since \( (x, z + t(y - z)) \) is in \( b(E) \) for each \( t \) in \([0, 1]\).
Choose a neighborhood $W$ of 0 in $X$ so that $h(W)$ is contained in $F \times G$. If each of $x$ and $y$ is in $W$ and $f'(0)(f(x)) = f'(0)(f(y))$ then $f(x) = f(h^{-1}(h(x))) = f(h^{-1}(h(y))) = f(y)$ so $(f'(0)|f(W))$ is one to one.

The idea for the last part of this proof is contained in Dieudonné's proof of the rank theorem [1, p. 277].

**Theorem 5.** Suppose $f$ satisfies Condition I. There is a neighborhood $S$ of 0 in $X$ and a continuously differentiable function $q$ with domain $S$ so that $q(S)$ is a neighborhood of 0 in $f^{-1}(0)$, $q \circ q = q$, and $(f \times q|S)$ is a homeomorphism onto a neighborhood of $(0, 0)$ in $f(S) \times f^{-1}(0)$.

**Proof.** By Conditions I and II we may choose a neighborhood $W$ of 0 in $X$ so that $(f'(0)|f(W))$ is one to one and if $V$ is a neighborhood of 0 in $N$ contained in $W$ then $f(V)$ is a neighborhood of 0 in $f(W)$. Choose neighborhoods $A$ and $B$ of 0 in $N$ so that $(f'(0)|/A)$ is a homeomorphism onto $B$, $(f'(0)|/A)^{-1}$ is continuously differentiable on $B$, and $A$ is contained in $W$. Choose $C$ and $D$ neighborhoods of 0 in $X$ so that each of $C$ and $D$ is contained in $W$, $f(C)$ is contained in $f(A)$, $(K_1|C)$ is a homeomorphism onto $D$, and $(K_1|C)^{-1}$ is continuously differentiable on $D$.

$f(A)$ is a neighborhood of 0 in $f(W)$ so $f(A) \times (I - f'(0))(W)$ is a neighborhood of $(0, 0)$ in $f(W) \times (I - f'(0))(W)$. Suppose each of $(a, b)$ and $(c, d)$ is in $f(A) \times (I - f'(0))(W)$ and $a + b = c + d$. There are $x$ and $y$ in $A$ so that $f(x) = a$ and $f(y) = c$. $d - b$ is in $\ker(f'(0))$ so $f'(0)(f(x)) = f'(0)(f(y))$. Thus $x = y$, $a = c$, and $b = d$ so $(f(A) \times (I - f'(0))(W))$ is one to one.

Let $g$ be defined on $C$ by $g(x) = (f(x), (I - f'(0))(x))$. Then $(K_1|C) = (+ \circ g|C)$. Hence, $(g|C)$ is one to one and $+(g(C)) = D$. $E = +^{-1}(D) \cap [f(A) \times (I - f'(0))(W)]$ is a neighborhood of $(0, 0)$ in $f(W) \times (I - f'(0))(W)$. Suppose $e$ is in $E$. $+(e) = K_1(z)$ for some $z$ in $C$ so $+(e) = +(g(z))$. But each of $e$ and $g(z)$ is in $f(A) \times (I - f'(0))(W)$ so $g(z) = e$. Thus $g(C)$ contains $E$.

Choose $F$ and $G$ neighborhoods of 0 in $f(W)$ and $(I - f'(0))(W)$, respectively, so that $F \times G$ is contained in $g(C)$. Let $h = (K_1^{-1}|G)$, and define $g$ on $(I - f'(0))^{-1}(G)$ by $g(x) = h((I - f'(0))(x)) = g^{-1}(0, x - f'(0)(x))$.

If $x$ is in $(I - f'(0))^{-1}(G)$ then $g(q(x)) = g(g^{-1}(0, x - f'(0)(x))) = (0, x - f'(0)(x))$. 

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Thus $q(x)$ is in $f^{-1}(0) \cap (I - f'(0))^{-1}(G)$.

$$q(q(x)) = g^{-1}(0, q(x) - f'(0)(q(x))) = g^{-1}(q(q(x))) = q(x)$$

if $x$ is in $(I - f'(0))^{-1}(G)$ so $q \circ q = q$.

Suppose $x$ is in $g^{-1}(F \times G) \cap f^{-1}(0)$. Then

$$x = g^{-1}(g(x)) = g^{-1}(0, x - f'(0)(x)) = q(x - f'(0)(x)).$$

Thus $\operatorname{im}(q)$ is a neighborhood of 0 in $f^{-1}(0)$.

Since $h = (K^{-1}_1|G) h$ is continuously differentiable on $G$, $q$ is continuously differentiable on $(I - f'(0))^{-1}(G)$. Moreover, $q'(0) = h'(0) \circ (I - f'(0))$, $h'(0) = (K^{-1}_1) \circ (I - f'(0))^{-1}$, $\operatorname{im}(I - f'(0)) = (I - f'(0))\operatorname{im}(I - f'(0))$ so $q'(0) = I - f'(0)$. From Lemma 3 of [2] there is a neighborhood $H$ of 0 in $\operatorname{im}(q'(0))$ so that $(q|H)$ is a homeomorphism onto a neighborhood of 0 in $\operatorname{im}(q)$. Choose $S$ a neighborhood of 0 in $X$ so that $S$ is contained in $(I - f'(0))^{-1}(H \cap G)$ and $S \cap N$ is contained in $A$. Define $r$ on $(S) \times (I - f'(0))(S)$ by $r(x, y) = (x, q(y))$. By choice of $H$, $r$ is a homeomorphism onto a neighborhood of 0 in $(S) \times q(S)$. $(f \times q|S) = (r \circ g|S)$. Thus $(f \times q|S)$ is a homeomorphism onto a neighborhood of $(0, 0)$ in $(S) \times q(S)$. This concludes the proof of Theorem 5.

We will now indicate an application of the preceding theorems to the local theory of differentiable semigroups. Suppose $D$ is an open set of $X$ containing 0, and $V$ is a continuously differentiable associative function from $D \times D$ into $X$ so that $V(0, 0) = 0$. Define $f$ on $D$ by $f(x) = V(x, 0)$. If each of $x$ and $f(x)$ is in $D$ then $f(f(x)) = f(x)$. Thus, by Theorem 2, $f$ satisfies Condition I. By Theorem 5 then there is a neighborhood $S$ of 0 in $X$ and a continuously differentiable function $q$ defined on $S$ so that $q \circ q = q$ and $\operatorname{im}(q)$ is a neighborhood of 0 in $f^{-1}(0)$. By Lemma 3 of [2] there is a neighborhood $V$ of 0 in $Y = \operatorname{im}(q'(0)) = \operatorname{im}(I - f'(0))$ so that $(q|V)$ is a homeomorphism onto a neighborhood of 0 in $\operatorname{im}(q)$, $(q'(0) \circ q|V)$ is a homeomorphism onto a neighborhood of 0 in $Y$, and $(q'(0) \circ q|V)^{-1}$ is continuously differentiable. Define $W$ contained in $(Y \times Y) \times Y$ by

$$W(x, y) = (q|V)^{-1}(V(q(x), q(y)))$$

whenever $V(q(x), q(y))$ is in $q(V)$. The domain of $W$ is a neighborhood of $(0, 0)$ in $Y \times Y$, $W$ is associative,

$$W(x, y) = (q|V)^{-1} \circ (q'(0)|q(V))^{-1} \circ q'(0) \circ V(q(x), q(y))$$

$$= (q'(0) \circ q|V)^{-1} \circ q'(0) \circ V(q(x), q(y))$$

so $W$ is continuously differentiable on a neighborhood of $(0, 0)$, and $q$ is a
local isomorphism between the local differentiable semigroups $((/^{-1}(\{0\}), (V)\backslash f^{-1}(\{0\}) \times f^{-1}(\{0\})))$ and $(Y, W)$.

In [3] it was shown that $\text{im}(f)$ is locally the topological and algebraic product of a local Lie group and a left trivial semigroup. Horne in [4] began a study of differentiable semigroups with right zero. The above shows that $(X, V)$ is, near $0$, the topological product of the differentiable subsemigroups $f(D)$ and $f^{-1}(\{0\})$.

Can $V$ be reconstructed from its restriction to $(f(D) \times f(D)) \cup (f^{-1}(\{0\}) \times f^{-1}(\{0\}))$?

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36830