COMPLETE INTERSECTION MANIFOLDS WITH EXTREMAL EULER-POINCARÉ CHARACTERISTICS

BANG-YEN CHEN1 AND KOICHI OGIUE2

ABSTRACT. Studies on n-dimensional complete intersection manifolds with Euler-Poincaré characteristic $n + 1$.

1. Introduction. Let $P_{n+p}(C)$ be an $(n + p)$-dimensional complex projective space. An $n$-dimensional algebraic manifold imbedded in $P_{n+p}(C)$ is called a complete intersection manifold if $M$ is given as an intersection of $p$ nonsingular hypersurfaces $M_1, \ldots, M_p$ in general position in $P_{n+p}(C): M = M_1 \cap \cdots \cap M_p$.

Let $b_i(M)$ be the $i$th Betti number of $M$. It is known that if $M$ is an $n$-dimensional complete intersection manifold, then

$$b_{2k}(M) = 1 \quad (2k \neq n),$$

$$b_{2k+1}(M) = 0 \quad (2k + 1 \neq n).$$

Therefore the Euler-Poincaré characteristic $\chi(M)$ of $M$ satisfies $\chi(M) \geq n + 1$ (resp. $\leq n + 1$) provided that $n$ is even (resp. odd). If $M$ is a linear subspace or an odd-dimensional hyperquadric, then the equality holds.

The purpose of this paper is to consider the converse.

2. Preliminaries. Let $P_{n+p}(C)$ be an $(n + p)$-dimensional complex projective space and let $M$ be an $n$-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$: $M = M_1 \cap \cdots \cap M_p$, where $M_\alpha$'s are nonsingular hypersurfaces in $P_{n+p}(C)$. The following result gives a concrete formula for the Euler-Poincaré characteristic of complete intersection manifold.

Proposition 1 [1]. Let $M = M_1 \cap \cdots \cap M_p$ be an $n$-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. If $\deg M_\alpha = a_\alpha$, then the Euler-Poincaré characteristic $\chi(M)$ of $M$ is given by
\[ \chi(M) = \left[ \sum_{k=0}^{n} (-1)^k \binom{n+p+1}{n-k} \sigma_k \right] \left( \prod_{a=1}^{p} a_a \right), \]

where \( \sigma_k = \sum_{a_1 \leq \cdots \leq a_k} a_{a_1} \cdots a_{a_k} \) (the sum of all homogeneous monomials of degree \( k \) in \( a_1, a_2, \cdots, a_p \)) and \( \binom{n+p+1}{n-k} \) is the binomial coefficient.

The following is an immediate consequence of Proposition 1 (cf. [1, Theorem 3.1]).

**Lemma 1.** Let \( M \) be an \( n \)-dimensional complete intersection manifold. If \( \chi(M) = \nu_1 \cdots \nu_p \) for some prime numbers \( \nu_1, \cdots, \nu_p \) (\( \neq \pm 1 \)), then \( M \) can be imbedded as a complete intersection manifold in \( \mathbb{P}^{n+p}(\mathbb{C}) \).

The smallest codimension for a complete intersection manifold is roughly estimated by the Euler-Poincaré characteristic.

**Theorem 1.** Let \( p \) be the smallest integer for which an \( n \)-dimensional complex manifold \( M \) can be imbedded as a complete intersection manifold in \( \mathbb{P}^{n+p}(\mathbb{C}) \). Then \( p \leq \log_2 |\chi(M)| \).

**Proof.** The assumption implies that \( \sigma_a \geq 2 \) for \( a = 1, \cdots, p \). Therefore it follows from Proposition 1 that \( |\chi(M)| \geq 2^p \), i.e., \( p \leq \log_2 |\chi(M)| \). Q.E.D.

The following is an answer to our problem in a special case.

**Theorem 2.** Let \( M = M_1 \cap \cdots \cap M_p \) be an \( n \)-dimensional complete intersection manifold imbedded in \( \mathbb{P}^{n+p}(\mathbb{C}) \). If \( n \) is even, \( \chi(M) = n+1 \) and \( \deg M_1 = \cdots = \deg M_p \), then \( M \) is a linear subspace.

**Proof.** Let \( \deg M_1 = \cdots = \deg M_p = a \). Then \( \sigma_k = \binom{p+k-1}{k} a^k \) so that

\[ \chi(M) = \frac{(n+p+1)!}{n!(p-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^k}{(p+k)(p+k+1)}. \]

Let \( \omega(a) \) denote the function of \( a \) defined on the interval \([0, \infty)\) and given by the right-hand side of the above equation. We put

\[ f(a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^k}{(p+k)(p+k+1)}, \quad g(a) = a f(a), \]

and

\[ h(a) = a g'(a) - g(a). \]

Then, by a straightforward computation, we have \( h'(a) = a^p (1-a)^n \), from
which it follows that \( h(a) > 0 \), i.e., \( ag'(a) - g(a) > 0 \) for all \( a > 0 \). Therefore \( f'(a) > 0 \) for all \( a > 0 \) so that the function \( \omega(a) \) as well as \( f(a) \) is monotonically increasing. Since \( \omega(1) = n + 1 \), it follows that \( \omega(a) > n + 1 \) for \( a > 1 \). Thus \( \chi(M) = n + 1 \) only when \( a = 1 \). Q.E.D.

Immediately from Theorem 2, we have

**Theorem 3.** Let \( p \) be the smallest integer for which an \( n \)-dimensional complex manifold \( M \) can be imbedded as a complete intersection manifold in \( P_{n+p}(C) \). If \( n \) is even and \( \chi(M) = n + 1 \), then \( p \leq 1 + \log_3 [(n + 1)/5] \).

3. Low codimensional case. The answer to the simplest case has been known as follows (cf. [3]).

**Proposition 2.** Let \( M \) be a compact hypersurface imbedded in \( P_{n+1}(C) \). If \( \chi(M) = n + 1 \), then \( M \) is either a hyperplane or a hyperquadric, the latter case arising only when \( n \) is odd.

Proposition 2, combined with Lemma 1, implies

**Theorem 4.** Let \( M \) be an \( n \)-dimensional complete intersection manifold imbedded in \( P_{n+p}(C) \), \( n > 1 \). If \( \chi(M) = n + 1 \) and if \( n + 1 \) is a prime number, then \( M \) is a linear subspace.

The following provides the answer to the case where codimension is 2.

**Theorem 5.** Let \( M \) be an \( n \)-dimensional complete intersection manifold imbedded in \( P_{n+2}(C) \). If \( \chi(M) = n + 1 \), then \( M \) is either a linear subspace or a hyperquadric in some hyperplane of \( P_{n+2}(C) \), the latter case arising only when \( n \) is odd.

**Proof.** Let \( M = M_1 \cap M_2 \) and \( a_\alpha = \deg M_\alpha \) (\( \alpha = 1, 2 \)). Then it follows from Proposition 1 that

\[
\chi(M) = \left[ \sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} \sigma_k \right] a_1 a_2.
\]

If \( a_1 = 1 \) or \( a_2 = 1 \), then Theorem 5 reduces to Proposition 2. Therefore it suffices to show that the Diophantine equation

\[
\left[ \sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} \sigma_k \right] a_1 a_2 = n + 1
\]

has no solution satisfying \( a_1 \geq 2 \) and \( a_2 \geq 2 \), which is equivalent to the non-existence of a solution to the Diophantine equation
Multiplying both sides of (1) by \( a_1 - a_2 \), we obtain

\[
\sum_{k=0}^{n} (-1)^k \binom{n+3}{n-k} \sigma_k = \xi \quad \text{for} \quad 0 < \xi \leq \frac{n+1}{4}.
\]

which can be written as

\[
(1-a_1)^{n+3} - 1 + (n+3)a_1/a_1^2 + \xi a_1 = \frac{(1-a_2)^{n+3} - 1 + (n+3)a_2}{a_2^2} + \xi a_2.
\]

Let

\[
f_n(a) = \frac{(1-a)^{n+3} - 1 + (n+3)a}{a^2 + \xi a}.
\]

Then it is not difficult to prove that

\[
f_n'(a) > 0 \quad \text{(resp.} < 0) \quad \text{for} \quad a \geq 2 \quad \text{if} \quad n \quad \text{is odd (resp. even).}
\]

This implies that \( f_n(a) \) is monotonically increasing (resp. decreasing) for \( a \geq 2 \) if \( n \) is odd (resp. even). Therefore from (2) we deduce that \( a_1 = a_2 \).

Putting \( a = a_1 = a_2 \), from (1) we have

\[
\sum_{k=0}^{n} (-1)^{k+1} \binom{n+3}{n-k} a^k = \xi.
\]

Let

\[
g_n(a) = \sum_{k=0}^{n} (-1)^{k+1} \binom{n+3}{n-k} a^k - \xi.
\]

Then we have \( f_n'(a) = -g_n(a) \), which, together with (3), implies that \( g_n(a) < 0 \) (resp. \( > 0 \)) for \( a \geq 2 \) if \( n \) is odd (resp. even). Therefore (4) has no solution satisfying \( a \geq 2 \).

Thus we have proved that (1) has no solution satisfying \( a_1 \geq 2 \) and \( a_2 \geq 2 \). Q.E.D.

As an immediate consequence of Theorem 5 and Lemma 1, we have

**Corollary.** Let \( M \) be an \( n \)-dimensional complete intersection manifold. If \( \chi(M) = n + 1 = \nu_1 \nu_2 \) for some prime numbers \( \nu_1 \) and \( \nu_2 \), then \( M \) is either a linear subspace or a hyperquadric in some \( (n+1) \)-dimensional linear subspace, the latter case arising only when \( n \) is odd.
4. Low dimensional case. In this section, we shall prove

**Theorem 6.** Let $M$ be an $n$-dimensional complete intersection manifold imbedded in $P_{n+p}(C)$. If $n \leq 52$ and $\chi(M) = n + 1$, then $M$ is either a linear subspace or a hyperquadric in some $(n + 1)$-dimensional linear subspace, the latter case arising only when $n$ is odd.

**Proof.** From Theorem 4 and the Corollary, we see that Theorem 6 holds except when $n$ is one of the following integers:

$$n = 7, 11, 15, 17, 19, 23, 26, 27, 29, 31, 35, 39, 41, 43, 44, 47, 49, 51.$$ Combining Theorems 2 and 5, we see that Theorem 6 is true for $n = 26$. For the cases where $n$ is one of the following:

$$(*) \quad n = 7, 11, 17, 19, 27, 29, 41, 43, 44, 49, 51,$$

we have $n + 1 = a_1 a_2 a_3$, where $a_1$, $a_2$ and $a_3$ are prime numbers. In all of these cases, if the smallest codimension is three, then we may assume that $\deg M_1 = a_1$, $\deg M_2 = a_2$ and $\deg M_3 = a_3$. By straightforward computations, we may easily find that

$$\sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n + 4 \\ n - k \end{array} \right) \sigma_k \neq 1.$$ Hence

$$\chi(M) = \left[ \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n + 4 \\ n - k \end{array} \right) \sigma_k \right] a_1 a_2 a_3 \neq n + 1.$$ This is a contradiction. Hence, for any integer $n$ given in $(*)$, the smallest codimension $p$ is $\leq 2$. Therefore, by Theorem 5, we see that Theorem 6 holds for all such $n$. For the remaining cases, $n = 15, 23, 35, 39$ and $47$, the smallest codimension $p$ is $\leq 4$ for $n = 15, 23, 35$ and $39$ and is $\leq 5$ for $n = 31$ and $47$. In all cases, by a straightforward computation for $\sum_{k=0}^{n} (-1)^k \left( \begin{array}{c} n + p + 1 \\ n - k \end{array} \right) \sigma_k$, we see that the only possibilities are either a linear subspace or a hyperquadric. Q.E.D.

Remark. From the results above, we would like to state the following

**Conjecture.** The only $n$-dimensional complete intersection manifolds $M$ in $P_{n+p}(C)$ with Euler-Poincaré characteristic $\chi(M) = n + 1$ are a linear subspace and a hyperquadric in some $(n + 1)$-dimensional linear subspace, the latter case arising only when $n$ is odd.
REFERENCES

