HOMOGENEOUS SPACES WITH VANISHING STEENROD SQUARING OPERATIONS

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ABSTRACT. If $G$ is a compact, connected Lie group, $H$ is a closed subgroup of $G$ and $G/H$ has no nonzero Steenrod operations, then $G/H$ splits as a product of homogeneous spaces of simple Lie groups (the factors of $G$). This fact is used to classify transitive actions on spaces with vanishing Steenrod operations, namely product of certain Stiefel manifolds and spheres.

1. Introduction. A homogeneous space $M$ is a differentiable manifold which admits a transitive, differentiable action by a compact, connected Lie group $G$. Any such $G$ may be uniquely expressed as $G = \overline{G}/N = \overline{G}_1 \times \cdots \times \overline{G}_s/N$ where $\overline{G}_i$ is either $S^1$ or a simple, simply connected Lie group and $N$ is a finite normal subgroup of $G$. If $H$ is an isotropy subgroup of the action of $G$ on $M$ and $\text{Rk}(H) = \text{Rk}(G)$, it is well known that $G = G_1 \times \cdots \times G_s$ ($G_i = \overline{G}_i/N_i, N_i = \overline{G}_i \cap N$) and $M$ is diffeomorphic to $G_1/H_1 \times \cdots \times G_s/H_s$ where $H_i = G_i \cap H$. If $\text{Rk}(G) > \text{Rk}(H)$, the above splitting may not occur. With the assumption that the action of $G$ is irreducible (no proper normal subgroup of $G$ acts transitively on $M$) the following results have been established: $M$ does split as a product if (1) $H^*(M; \mathbb{Q})$ is a Hopf algebra with some dimensional restrictions [3], or (2) $M$ is highly connected relative to $\text{Rk}(G) - \text{Rk}(H)$ [7]. In this paper we prove:

**Theorem 1.** Let $M$ be 11-connected and have all trivial mod-2 Steenrod squaring operations. If there is an irreducibly transitive, effective action of $G$ on $M$, $M$ is diffeomorphic to $G_1/H_1 \times \cdots \times G_s/H_s$ where the $G_i$ are simple factors of $G$ and $H_i = G_i \cap H$.

As in [3], [7] this splitting (or decomposition) theorem can be used to classify some homogeneous spaces.
Theorem 2. If \( M \) is an 11-connected homogeneous space with vanishing Steenrod squaring operations, then \( M \) can only be a product of spheres and the Stiefel manifolds \( V_{n,n-2} = \text{SO}(n)/\text{SO}(n-2) \), \( W_{n,n-2} = \text{SU}(n)/\text{SU}(n-2) \) and \( X_{n,n-2} = \text{Sp}(n)/\text{Sp}(n-2) \) with \( n \) even.

Theorem 3. If \( M \) is a homogeneous space which has the same homotopy type as a product of spheres, all of dimension greater than 11, then \( M \) is diffeomorphic to that product of standard spheres and any effective, irreducibly transitive action on \( M \) is a product of the known actions on each sphere.

The condition that \( M \) be 11-connected in Theorem 1 is necessary as the following example shows: Let \( G = \text{Sp}(3) \times \text{Sp}(n) \) with \( n \) even and \( n > 4 \). Now there are standard embeddings \( \phi_1: \text{Sp}(2) \to \text{Sp}(3) \) and \( \phi_2: \text{Sp}(2) \times \text{Sp}(n-2) \to \text{Sp}(n) \) so we can define \( \phi: \text{Sp}(2) \times \text{Sp}(n-2) \to \text{Sp}(2) \times \text{Sp}(n) \) by \( \phi(g_1, g_2) = (\phi_1(g_1), \phi_2(g_1, g_2)) \). If \( H \) is the image of \( \phi \), \( G/H \) is 10-connected, has all trivial Steenrod operations, and the action is not a product action. Theorem 3 extends Theorem B of [7] to a much larger class of products of spheres.

2. Preliminaries. We will consider only \( \mathbb{Z}_2 \)-cohomology and \( G \) will always act irreducibly. If \( G \) acts transitively and effectively on \( M \), then \( \overline{G} \) has a natural transitive, almost effective action on \( M \). Just as the action of \( G \) on \( M \) can be represented as translation on \( G/H \), the action of \( \overline{G} \) on \( M \) can be represented as translation on \( \overline{G}/H \).

The mod 2-cohomology of the simply connected, simple classical groups is as follows [2], [6]:

\[
H^\ast(\text{Spin}(n)) \text{ has a simple system of generators } w_3, \ldots, w_{n-1}, u, \text{ where }
\]
\[
dim w_i = i, \quad w_i \cup w_i = w_{2i} \text{ if } 2i \leq n-1,
\]
\[
dim u = t-1 \quad (t \text{ is the smallest power of } 2 \text{ larger than } n-1),
\]
\[
dim w_i = 0 \text{ if } i \text{ is a power of } 2, \text{ and }
\]
\[
\text{Sq}^i(w_i) = C_i^j w_{i+j}, \text{ if } i + j \leq n-1, \text{ where } C_i^j \text{ is the binomial coefficient reduced mod 2.}
\]

\[
H^\ast(\text{SU}(n)) \text{ and } H^\ast(\text{Sp}(n)) \text{ are exterior algebras with generators } w_{dj+d-1}, j = 1, \ldots, n-1 \text{ for } \text{SU}(n) \text{ and } j = 0, \ldots, n-1 \text{ for } \text{Sp}(n),
\]
\[
dim w_{dj+d-1} = dj + d - 1, \text{ where } d = 2 \text{ for } \text{SU}(n) \text{ and } d = 4 \text{ for } \text{Sp}(n), \text{ and }
\]
\[
\text{Sq}^i w_{dj+d-1} = C_i^j w_{dj+i} + d-1 \text{ for } j + i \leq n-1,
\]
\[
= 0 \text{ otherwise.}
\]

We organize some facts in the following:
Lemma 2.1. Let $\overline{G}/\overline{H}$ be 11-connected with $\overline{G}$ acting irreducibly. Then all the factors of $\overline{G}$ have rank 4 or greater and all the factors of $\overline{H}$ have rank 3 or greater. For each Spin, SU or Sp factor of $\overline{G}$, there is a corresponding Spin, SU or Sp factor of $\overline{H}$ which projects nontrivially onto it. Also the only possible exceptional factor of $\overline{G}$ is $E_8$ and then $\overline{H}$ must have a factor $E_7$ which projects into it.

Proof. By Lemmas 2.1 and 2.2 of [7], $\iota: H \rightarrow G$ induces an isomorphism through dim 10, in cohomology, and $\overline{G}$ and $\overline{H}$ have the same number of simple factors. Let $\overline{G}_i$ be a factor of $\overline{G}$ and $p_i: \overline{G} \rightarrow \overline{G}_i$ the projection. It was pointed out in [7] that the action of $\overline{G}$ is not irreducible if $p_i(\overline{G}) = \overline{G}_i$, and in the proof of [7, Theorem A] it was shown that $p_i(\overline{H}) = \overline{G}_i$ if $\overline{G}/\overline{H}$ is 11-connected and $\overline{G}_i$ has rank less than four. The $\mathbb{Z}_2$-structure of the exceptional groups can be found in [8]. Since $E_7$ is the only simple group with an indivisible generator in dim 4, $F_4$ is a factor of $\overline{G}$ if and only if $F_4$ is a factor of $\overline{H}$. But then the projection of $\overline{G}$ onto an $F_4$ factor would carry $\overline{H}$ onto that factor, which again is impossible since $\overline{G}$ acts irreducibly.

Since $E_6$, $E_7$ and $E_8$ all have an indivisible generator in dim 9 which is not the Steenrod square of a generator in dim 7 and the classical groups do not, $\overline{G}$ has an $E$ factor if and only if $\overline{H}$ does. Since $E_6$ has no generator in dim 10 and $E_7$ and $E_8$ do, the only possibility is that $\overline{G}$ may have an $E_8$ factor and then $\overline{H}$ must have an $E_7$ factor. For each Sp factor of $\overline{G}$ we have a generator $w_3$ of $H^3(\overline{G})$ such that $Sq^2w_3 = 0$ and $w_3 \cup w_3 = 0$. Hence the isomorphism $i^*$ must carry $w_3$ onto a generator (possibly modulo some other elements) with the same properties. Hence $\overline{H}$ must have an Sp factor which projects nontrivially onto the Sp factor of $\overline{G}$. For the SU case, we have a generator $w_3$ such that $Sq^2w_3 \neq 0$ and $w_3 \cup w_3 = 0$ and for the Spin case we have a $w_3$ with $Sq^2w_3 \neq 0$ and $w_3 \cup w_3 \neq 0$, so these cases are handled similarly. Since $\overline{G}/\overline{H}$ is 11-connected, $i^*$ is also injective in dim 11. Noting the restrictions on the factors of $\overline{G}$ and the cohomology of the classical groups listed above, we see that the factors of $\overline{H}$ must have rank at least three.

3. Proofs. Suppose $M$ (or $\overline{G}/\overline{H}$) does not split as a product. Let $q$ be the first dimension past zero where $H^q(M) \neq 0$. By Lemma 2.1 of [7], $i: \overline{H} \rightarrow \overline{G}$ induces an isomorphism through dim $q - 2$ and $i^*$ is injective in dim $q - 1$. If $q$ is a power of two and Spin $(q)$ is a factor of $\overline{H}$, we have the "extra" generator $u \in H^{q-1}(\text{Spin}(q))$ and so it is possible that dim $H^{q-1}(\overline{G}) <$ dim $H^{q-1}(\overline{H})$. If, however, $q$ is not a power of two, there are no "extra"
generators in dim $q - 1$; and, since $\overline{G}$ and $\overline{H}$ have the same number of factors, dim $H^{q-1}(\overline{G}) = \dim H^{q-1}(\overline{H})$.

We first consider the case when $q$ is not a power of two; i.e., when $i^*$ is an isomorphism through dim $q - 1$. Hence $p^*: H^q(M) \to H^q(\overline{G})$ is injective where $p: \overline{G} \to M$ is the projection of the bundle $\overline{H} \to \overline{G} \to M$. If $0 \neq v \in H^q(M)$, $p^*(v)$ cannot be a divisible element in $H^q(\overline{G})$ because this would imply that $i^* p^*(v) \neq 0$. So, if $j: \overline{G} \to \overline{H}$ is inclusion as a factor, $j^* p^*(v)$ must be an indivisible element (possibly modulo some divisible elements) in the cohomology of one of the factors of $\overline{G}$.

Let us first consider the case when $i^* p^*(v)$ is nonzero in a Spin factor. If $q$ is odd and $q + 1$ is not a power of two, $Sq^1 w = w_{q+1}$ in $H^*(\text{Spin})$. But then by naturality of the Steenrod squares, $Sq^1(v)$ is nonzero unless $w_{q+1} = 0$. This means that the Spin factor in question can only be Spin$(q + 1)$. If $q$ is odd and $q + 1$ is a power of two, $Sq^2 w = w_{q+2}$. We also have $Sq^2 w = w_{q+2}$ if $q$ is even, not divisible by four and $q + 2$ is not a power of two. Hence $w_{q+2} = 0$ (for otherwise $Sq^2 v \neq 0$ by naturality) and the Spin factor can only be Spin$(q + 1)$ or Spin$(q + 2)$. If $q$ is even, not divisible by four and $q + 2$ is a power of two, $Sq^4 w = w_{q+4}$ and we get Spin$(q + 3)$ and Spin$(q + 4)$ as possible factors. The case where $q$ is divisible by four is technically more difficult. Let Spin$(n)$ be the factor and consider $w_{q-1}, w_q, w_{q+1} \in H^*(\text{Spin}(n))$. Suppose $i^* (w_{q+1}) \neq 0$ in $H^*(\overline{H})$. (We identify the $w$'s with elements in $H^*(\overline{G})$.) Since $Sq^2 w_{q-1} = w_{q+1}$, $Sq^2(i^* (w_{q-1})) = i^* w_{q+1}$. Hence $i^*(w_{q-1})$ and $i^*(w_{q+1})$ must project into the same factor of $\overline{H}$. Since $Sq^1 w_{q-1} = w_q$ and $i^* w_q = 0$, $Sq^1(i^* (w_{q-1})) = 0$; this means that the factor of $\overline{H}$ is not a Spin factor. Since the generators $i^*(w_{q-1})$ and $i^*(w_{q+1})$ are two dimensions apart, the factor cannot be an Sp. So it must be an SU factor and must in fact be at least SU($(q + 2)/2)$. By Lemma 2.1, $\overline{H}$ has a Spin factor which projects into Spin$(n)$. Since $i^*$ is an isomorphism through dim $q - 1$, this Spin factor must be at least a Spin$(q)$. So the image of $\overline{H}$ under the projection into Spin$(n)$ contains a subgroup locally isomorphic to Spin$(q) \times SU((q + 2)/2)$. Now let $q = kt$ where $k$ is odd and $t$ is a power of two. Since $C^q_{q-t} \neq 0 \mod 2$, we have $Sq^t w_q = w_{q+t}$ if $q + t$ is not a power of two and $Sq^t w_q = w_{2q-t}$ if $q + t$ is a power of two. So as before $w_{q+t}$ or $w_{2q-t}$ must be zero respectively. This gives the restriction $n \leq \max(q + t, 2q - t) \leq 2q - 4$. But

$$\mathrm{Rk} \left( \text{Spin}(q) \times SU \left( \frac{q + 2}{2} \right) \right) = \frac{q}{2} + \frac{q}{2} > q - 2 \geq \mathrm{Rk} (\text{Spin}(n)).$$
Faced with this contradiction, we conclude that \( i^*(w_{q+1}) = 0 \). But now if we examine the spectral sequence for \( \overline{H} \to \overline{G} \to M \), we see that there must be a nonzero element \( z \in H^{q+1}(M) \) such that \( j^*p^*(z) = w_{q+1} \), modulo some divisible elements. But \( Sq^1(w_{q+1}) = w_{q+2} \), hence \( w_{q+2} \) must be zero and \( n = q + 1 \) or \( q + 2 \). In summary, we have established that if \( j^*p^*(v) \) is nonzero in a Spin factor with \( q \) not a power of two, the factor is \( \text{Spin}(q+i) \), \( i = 1, 2, 3, 4 \). Also, since \( i^* \) is an isomorphism through \( \dim q - 1 \), any Spin factor of \( \overline{H} \) must be at least a \( \text{Spin}(q) \). By Lemma 2.1, a \( \text{Spin}(q) \) factor of \( \overline{H} \) must project nontrivially into \( \text{Spin}(q+i) \); and, since all factors of \( \overline{H} \) have rank three or greater, the image of \( \overline{H} \) has no other factor.

The cases where \( j^*p^*(v) \) is nonzero in an SU or Sp factor are similar. Write \( q = dk + d - 1 \) where \( d = 2 \) or \( 4 \) if \( j^*p^*(v) \) is nonzero in an SU or Sp factor, respectively. The argument given above is repeated using \( Sq^d w_q \) if \( k \) is odd, \( Sq^{d+2} w_q \) if \( k \) is even but not divisible by four and \( Sq^{d+2} w_{q+d} \) if \( k \) is divisible by four. No special care has to be taken if \( k, k + 1 \) or \( k + 2 \) is a power of two. Hence we get only \( \text{SU}((q+1)/2), \text{SU}((q+3)/2), \text{Sp}((q+1)/4) \) or \( \text{Sp}((q+5)/4) \) as possible factors of \( \overline{G} \) and \( \text{SU}((q-1)/2) \) or \( \text{Sp}((q-3)/4) \) as possible factors of \( \overline{H} \) which project nontrivially into the SU or Sp factors, respectively.

By Lemma 2.1, \( E_8 \) is the only possible factor of \( \overline{G} \) and \( E_7 \) must project into it. Using the information listed in [8], we see that an isomorphism from \( H^*(E_8) \) to \( H^*(E_7) \) through \( \dim 10 \) must extend to an isomorphism through \( \dim 14 \). But \( \dim H^{15}(E_8) = \dim H^{15}(E_7) + 1 \). If \( q = 15 \) and \( j^*p^*(v) \) is nonzero in \( H^{15}(E_8) \), it can only map onto elements in \( H^{15}(E_8) \) which have nonzero Steenrod squares. Thus the exceptional groups may be eliminated.

Now we must dispose of the case when \( q \) is a power of two. Since \( H^q(\overline{G}) \) has only divisible elements, \( p^*(v) = 0 \) for \( v \in H^q(M) \). Also \( i^* \) is an isomorphism through \( \dim q - 2 \) and injective in \( \dim q - 1 \). Studying the spectral sequence, we see that this is only possible if \( \dim H^{q-1}(\overline{G}) < \dim H^{q-1}(\overline{H}) \) and the differential \( d^q_{0,q-1}: H^{q-1}(\overline{H}) \to H^q(M) \) is onto. This implies that one of the factors of \( \overline{H} \) must be \( \text{Spin}(q) \) as checking the cohomology listed above will show. But now with the same argument we used when \( q \) was divisible by four, we see that there is a nonzero element \( z \in H^{q+1}(M) \) such that \( j^*p^*(z) = w_{q+1} \), modulo some divisible elements, in \( H^*(\text{Spin}) \). But \( Sq^1 w_{q+1} = w_{q+2} \), so the Spin factor must again be \( \text{Spin}(q+1) \) or \( \text{Spin}(q+2) \).

Rewrite \( \overline{G} = G_i \times \overline{G}_i \), where \( G_i \) is the factor of \( \overline{G} \) selected above, and
let $H_1$ be the factor of $\overline{H}$ which projects into $G_1$. Let $\Gamma_1$ be the image of the homomorphism obtained by composing the inclusion $H_1 \rightarrow \overline{H}$ and the projection $\overline{H} \rightarrow G_1$. This homomorphism must be standard since $H_1$ maps onto a large subgroup of $G_1$; for example, if $H_1$ is $\text{Sp}(k)$, $G_1$ is $\text{Sp}(k + 1)$ or $\text{Sp}(k + 2)$. Hence $G_1/\Gamma_1$ is a standard Stiefel manifold. Let $\overline{G}_1 = \overline{G}_1 \cap \overline{H}$. We now consider the fiber bundle $\overline{G}_1/\overline{H}_1 \rightarrow \overline{M} \rightarrow \overline{G}_1/\overline{\Gamma}_1$ and claim that $\pi_1^*$ is injective. Projection on the factor $G_1$ gives the following map of bundles:

$$
\begin{array}{ccc}
\overline{H} & \rightarrow & \overline{G} \\
\downarrow & & \downarrow \pi_1 \\
\Gamma_1 & \rightarrow & G_1/\Gamma_1
\end{array}
$$

If $G_1/\Gamma_1$ is a complex or quaternionic Stiefel manifold, $p_1^*$ and hence $\pi_1^*$ is injective. In the real case $\text{Spin}(q + i)/\text{Spin}(q)$, if $q$ is not a power of two and $i = 1$ or if $q$ and $q + 1$ are not a power of two and $i = 2$, $p_i^*$ is also injective. The other real cases when $q$, $q + 1$ or $q + 2$ is a power of two are more troublesome because of the "extra" generator $u \in H^*(\text{Spin})$. Identify $u$ with an element in $H^*(\overline{H})$. By using the Steenrod squares of the classical groups we see that $u \not\in \text{Image } i^*$. Hence the differential map $d_i^{r+1}: H^r(\overline{H}) \rightarrow H^{r+1}(M)$ must take $u$ to a nonzero element for $r + 1 = q$, $q + 1$, or $q + 2$. Suppose $q + 1$ is a power of two and we are studying the case $\text{Spin}(q + 2)/\text{Spin}(q)$. The cohomology of this Stiefel manifold is an exterior algebra on generators $v$, $v_i$, $v_{q+1}$ with $p_i^*v_q = w_q$ and $d^{q+1}_0(u) = v_{q+1}$. Since the homomorphisms must commute with the differentials, $\pi_1^*v_{q+1} \neq 0$. The other cases are similar.

Even though $p_1^*$ is injective, $i_1^*$ need not be surjective. If $q \geq q$ is the first nontrivial positive dimension of $H^*(\overline{G}_1/\overline{H}_1)$, then from the spectral sequence we see that the first nonzero differential cannot occur until dimension $q_1 + q - 1 \geq q_1 + 11$. So $i_1^*$ is surjective through dimension $q_1 + q - 2 \geq q_1 + 10$ and so $H^*(\overline{G}_1/\overline{H}_1)$ has no nonzero squaring operations up to that dimension. This restriction is sufficient to repeat our procedure on $\overline{G}_1/\overline{H}_1$. Let us also note that the factor $H_1$ is not contained in $G_1$. If it were, $M = G_1/\overline{H}_1 \times \overline{G}_1/\overline{H}_1$ which is contrary to our assumption. This means that $H_1$ must project nontrivially into $\overline{G}_1$ and so $\text{Rk}(\overline{G}_1) - \text{Rk}(\overline{H}_1) \geq 3$ by Lemma 2.1.

After using our procedure on $\overline{G}_1/\overline{H}_1$ we get $\overline{G}_1 = G_2 \times \overline{G}_2$ and the fiber
bundle $\overline{G_2/H_2} \rightarrow \overline{G_1/H_1} \rightarrow G_2/\Gamma_2$ where the groups indexed by two have the appropriate definition and the projection induces an injection on cohomology. Now we have the bundle

$$\overline{G_2/H_2} \xrightarrow{i_2} M \xrightarrow{\pi_2} G_1/\Gamma_1 \times G_2/\Gamma_2$$

and we see that $\pi_2^*$ is injective because of the following diagram:

Let $q_2 \geq q_1$ be the first positive, nontrivial dimension of $H^*(\overline{G_2/H_2})$. As before, $i_2^*$ is surjective through dimension $q_2 + 10$ (and so the procedure may be used again on $\overline{G_2/H_2}$ and $\text{Rk}(\overline{G_2}) - \text{Rk}(\overline{H_2}) \geq 3$ for otherwise $M$ splits as a product.

We continue until we get the bundle

$$\overline{G_s/H_s} \xrightarrow{i_s} M \xrightarrow{\pi_s-1} G_1/\Gamma_1 \times \cdots \times G_{s-1}/\Gamma_{s-1}$$

with $\overline{G_s}$ and $\overline{H_s}$ simple, $\text{Rk}(\overline{G_s}) - \text{Rk}(\overline{H_s}) \geq 3$ and no Steenrod operations through dimension $q_s + 10$. But now we reach a contradiction, for the only possibilities for $\overline{G_s}$ and $\overline{H_s}$ as listed above have rank difference less than three. Hence we must conclude that $M$ does split as a product and we get that $M$ is diffeomorphic to $\overline{G_1/H_1} \times \cdots \times \overline{G_s/H_s}$ where $H_i = G_i \cap H$. Since $N = N_1 \times \cdots \times N_s$ where $N_i = \overline{G_i} \cap N$, $G = \overline{G}/N = \overline{G_1}/N_1 \times \cdots \times \overline{G_s}/N_s = G_1 \times \cdots \times G_s$ and so $M$ is diffeomorphic to $G_1/H_1 \times \cdots \times G_s/H_s$, $H_i = G_i \cap H$. This proves Theorem 1.

Since the factors $G_i/H_i$ must have all trivial Steenrod operations and $G_i$ and $H_i$ are simple, $G_i/H_i$ can only be $\text{SO}(n)/\text{SO}(n - 1) = S^{n-1}$, $\text{SO}(n)/\text{SO}(n - 2) = V_{n,n-2}$ if $n$ is even, $\text{SU}(n)/\text{SU}(n - 1) = S^{2n-1}$, $\text{SU}(n)/\text{SU}(n - 2) = W_{n,n-2}$ if $n$ is even, $\text{Sp}(n)/\text{Sp}(n - 1) = S^{4n-1}$, or $\text{Sp}(n)/\text{Sp}(n - 2) = X_{n,n-2}$ if $n$ is even. This establishes Theorem 2.

If $M$ is a homotopy product of spheres, $V_{n,n-2}$, $W_{n,n-2}$ and $X_{n,n-2}$ cannot be a factor of $M$ by Lemma 2.3 of [7] since these spaces are not homotopy products of spheres by [3]. Hence each factor $G_i/H_i$ is a homotopy sphere. By the work of Montgomery and Samelson [4], Borel [1] and Poncet [5], each $G_i/H_i$ is a standard sphere and all the transitive actions have been classified. This means that $M$ is a product of standard spheres and if
G has an irreducibly transitive action on M it is a product of the known actions on each sphere.

REFERENCES