DEFINING NORMAL SUBGROUPS
OF UNIPOTENT ALGEBRAIC GROUPS

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ABSTRACT. Let $G$ be a connected unipotent algebraic group defined over the perfect field $k$. We show that polynomial generators $x_1, \ldots, x_n$ for the ring $k[G]$ can be chosen so that if $N$ is any connected normal $k$-closed subgroup of $G$, then $I(N)$ can be generated by codim $N$ $p$-polynomials in $x_1, \ldots, x_n$ where $p = \text{char } k$. Moreover $k[G/N]$ can also be generated as a polynomial algebra over $k$ by $p$-polynomials.

Introduction. These results are essentially an extension of a theorem of Rosenlicht [4, Theorem 1].

We use the notation and conventions of [1] throughout this paper.

Recall that a $p$-polynomial in $k[T]$ is a linear form if $p = 0$ and a polynomial all of whose exponents are powers of $p$ if $p > 0$. A $p$-polynomial in $k[x_1, \ldots, x_n]$ is a sum of $p$-polynomials in each of the single variables $x_1, \ldots, x_n$. A function $f \in k[G]$ will be called additive if $f(ab) = f(a) + f(b)$ for all closed points $a, b$ in $G$.

1. Frattini coordinates. Let $G$ be a unipotent algebraic group. The Frattini subgroup of $G$ is the intersection of all closed subgroups of codimension one. We shall denote this group by $Fr(G)$.

Proposition 1. If $G$ is a unipotent algebraic group then $Fr(G)$ is a closed characteristic subgroup of $G$. If $G$ is connected and defined over the perfect field $k$, then $Fr(G)$ is connected and defined over $k$. Moreover in the connected case $G/Fr(G)$ has the structure of a vector group (over $k$ if $G$ is defined over $k$) and is the maximal such quotient.

Proof. The first assertion is immediate. Let $H \subset G$ be a connected subgroup of $G$ of codimension one. Since $G/H \simeq G_a$, $H$ contains the commutator subgroup of $G$ and the group generated by the $p$th powers of the elements of $G$. It follows that $Fr(G)$ also contains these subgroups.

Thus $G/\text{Fr}(G)$ is connected, commutative and of exponent $p$ hence by
[3, Proposition 2] has the structure of a vector group. If $N \subset G$ is any normal subgroup such that $G/N$ is isomorphic to $G^r_a$ for some integer $r$, then consider the natural map $G \to G/N \cong G^r_a$ followed by projection $\Pi_i$ onto each factor, $i = 1, 2, \ldots, r$. Each $\Pi_i$ is a homomorphism with kernel say $H_i$ and $\bigcap H_i = N$. Since $H_i$ has codimension one, $N \supset Fr(G)$ and $G/N$ is an image of $G/Fr(G)$ which establishes the maximality assertion.

As for rationality and connectedness, let $N$ be the closed normal subgroup generated by the commutator subgroup and $p$th powers of the elements of $G$. Then $N \subset Fr(G)$, $N$ is connected and $G/N$ has the structure of a vector group [3, Proposition 2] so $N = Fr(G)$. Since $N$ is defined over $k$ so is $G/N$ [1, 6.8]. This completes the proof.

Now let $G$ be a connected unipotent algebraic group and $Fr(G)$ the Frattini subgroup of $G$. If $G$ is defined over the perfect field $k$ then by [4, Corollary 2 of Theorem 1], $k[G]$ is $k$-isomorphic to $k[G/Fr(G)] \otimes k[Fr(G)]$. Let $x_1, \ldots, x_r$ be additive coordinates for the vector $k$-group $G/Fr(G)$ (cf. [3, §1]). Then $k[G] = k[x_1, \ldots, x_r] \otimes k[Fr(G)]$. By the proposition $F_1 = Fr(G)$ is again connected and defined over $k$ and we may continue this process until we arrive at a complete set of polynomial generators for $k[G]$. A set of polynomial generators $x_1, \ldots, x_n$ obtained in this way will be called a set of Frattini coordinates for $G$.

In case $G$ itself has the structure of a vector group, these coordinates have essentially been studied by Rosenlicht [3], [4] and Tits [7, III, 3.3]. In particular the following proposition is easily deduced from their results.

**Proposition 2.** Let $V$ be a connected unipotent algebraic group defined over the perfect field $k$. Suppose $V$ has the structure of a vector group over $k$ and $x_1, \ldots, x_n$ are Frattini coordinates for $V$. Then

(i) if $W$ is any $k$-closed subgroup of $V$ then $1(W)$ is generated by codim $W$ $p$-polynomials in $x_1, \ldots, x_n$;

(ii) the Frattini coordinates of $k[V/W] \subset k[V]$ are $p$-polynomials in the Frattini coordinates of $V$.

Now let $G$ be any connected unipotent group defined over the perfect field $k$. Let $N \subset Fr(G) = F$ be a $k$-closed normal subgroup of $G$. Then since $G/N \cong G/F \times F/N$ we have $k[G/N] \cong k[G/F] \otimes k[F/N]$. It follows from (ii) above that if $Fr(F) = e$ then a set of Frattini coordinates for $G/N$ may be taken to be $p$-polynomials in any fixed set of Frattini coordinates of $G$.

**Theorem.** Let the connected unipotent algebraic group $G$ be defined over the perfect field $k$. Let $x_1, \ldots, x_n$ be a fixed set of Frattini coordi-
nates for G. Suppose \( Z \) is a closed connected central one dimensional subgroup of \( G \) defined over \( k \). Then there exists a set of Frattini coordinates of \( G/Z \) in \( R = k[G/Z] \subset k[G] \) which consists of \( p \)-polynomials in \( x_1, \ldots, x_n \).

**Proof.** Let \( F_0 = G \supset F_1 = Fr(G) \supset \cdots \supset F_s \supset e \) be the Frattini series of \( G \). We argue by induction on the length, \( s \), of the series. Thus, suppose \( s = 1 \). If \( Z \subset F_1 \) then \( G/Z \cong G/F \times F/Z \), hence \( k[G/Z] \cong k[G/F] \otimes k[F/Z] \).

But by the remarks above, \( k[G/Z] \) has \( p \)-polynomials in \( x_1, \ldots, x_n \) as Frattini coordinates.

If \( Z \cap F_1 \) is finite we distinguish two cases.

Case 1. \( Z \cap F_1 = e \). Then \( ZF_1/F_1 \) is a direct factor of \( G/F_1 \) and is not equal to \( G/F_1 \) since \( Z \) is contained in a subgroup of codimension one.

Let \( L \supset F_1 \) be a connected \( k \)-closed subgroup of \( G \) such that \( L/F_1 \) is a complement of \( ZF_1/F_1 \) in \( G/F_1 \) [3, Proposition 1]. Then \( \text{codim} \ L = 1 \), hence \( L \) is normal in \( G \). If \( N = L \cap Z \) then \( NF_1/F_1 = e \), hence \( N \subset Z \cap F_1 \).

Thus \( L \cap Z = e \) and clearly \( LZ = G \).

Consider the commutative diagram
\[
\begin{array}{ccc}
G/F_1 & \overset{i}{\longrightarrow} & G \\
\downarrow{m} & & \downarrow{m'} \\
Z \times L & \xrightarrow{\nu} & ZF_1/F_1 \times L/F_1 \\
\end{array}
\]

where \( i \) is inclusion, \( \pi \) the quotient morphism, \( m \) and \( m' \) multiplication, and \( \nu = \pi|_{Z} \times \pi|_{L} \).

We obtain a commutative diagram of Lie algebras:
\[
\begin{array}{c}
\mathfrak{L}(F_1) \\
\downarrow{d_i} \\
\mathfrak{L}(G) \\
\downarrow{d \pi} \\
\mathfrak{L}(G/F_1) \to 0 \\
\end{array}
\begin{array}{c}
\mathfrak{L}(Z) \oplus \mathfrak{L}(L) \oplus \mathfrak{L}(ZF_1/F_1) \oplus \mathfrak{L}(L/F_1) \\
\downarrow{d \alpha} \\
\mathfrak{L}(F_1) \\
\downarrow{d m} \\
\mathfrak{L}(G) \\
\downarrow{d m'} \\
\mathfrak{L}(L) \\
\end{array}
\]

Since \( \alpha \) and \( dm' \) are surjective so is \( dm \). Thus \( dm \) is an isomorphism and \( m \) is separable. It follows from [1, Chapter II, 6.1] that \( m : Z \times L \to G \) is an isomorphism.

Now choose new Frattini coordinates \( y_1, \ldots, y_r, y_{r+1}, \ldots, y_n \) such that \( V(y_1, \ldots, y_r) = F_1 \) and \( V(y_1) = L \). Then \( y_1, \ldots, y_r \) are \( p \)-polynomials in \( x_1, \ldots, x_r \) and \( k[G/Z] = k[L] = k[y_2, \ldots, y_r, x_{r+1}, \ldots, x_n] \).

**Case 2.** \( \Lambda = Z \cap F \neq e \). Then in \( G/\Lambda \) we have the conditions of Case 1.
Hence \( k[G/Z] = k[G/A/Z/A] \subseteq k[G/A] \) is generated by \( p \)-polynomials in any set of Frattini coordinates of \( G/A \). But we may assume these last are \( p \)-polynomials in \( x_1, \ldots, x_n \). Hence \( k[G/Z] \) has a set of Frattini coordinates consisting of \( p \)-polynomials in \( x_1, \ldots, x_n \) and the case \( s = 1 \) is done.

If \( s > 1 \) we form the chain
\[
G \supset F'_1 \supset F_1 \supset F'_2 \supset F_2 \supset \cdots \supset F'_s \supset Z \supset e
\]
where \( F'_i/Z \) is the \( i \)th term in the Frattini series of \( G/Z \), and \( F'_i/Z \) has the structure of a vector group over \( k \). Each \( F'_i \) may be taken to be connected, closed and defined over \( k \).

Suppose \( l \geq 2 \). Then \( F_1 \supset Z \). By induction \( k[F_1/Z] \subseteq k[F_1] \) has a set of Frattini coordinates which consists of \( p \)-polynomials in \( x_1, \ldots, x_n \). It then follows as before from the isomorphism \( k[G/Z] \simeq k[G/F_1] \otimes k[F_1/Z] \) that \( G/Z \) has the desired property.

If \( l = 1 \) then \( F'_1/Z \) has the structure of a vector group. But then \( F_1 \supset F'_1 \) and if \( Z \subseteq F_1 \) we are done arguing as above. If not, \( F_1 \cap F_1 \) is finite and \( F_1 \rightarrow F'_1/Z \cap F_1 \) is an isogeny whose image is a vector group. Hence \( F_1 \) itself has the structure of a vector group so \( s = 1 \) a contradiction. This completes the proof.

**Corollary 1.** Let \( G \) be a connected unipotent group defined over the perfect field \( k \). Let \( N \) be a connected closed normal subgroup of \( G \) also defined over \( k \). Then the Frattini coordinates of \( G/N \) in \( k[G/N] \) can be taken to be \( p \)-polynomials in any fixed set of Frattini coordinates of \( G \).

**Proof.** Any connected closed subgroup normal in \( G \) and defined over \( k \) contains a central connected subgroup of dimension one defined over \( k \) by [5]. The corollary now follows by induction on the dimension of \( N \).

**Corollary 2.** Suppose \( G \) and \( k \) are as above. Then every normal closed connected subgroup \( N \) of \( G \) which is defined over \( k \) can be defined by \( d = \text{codim } N \) \( p \)-polynomials in \( x_1, \ldots, x_n \). Moreover these may be chosen so as to generate the ideal \( I(N) \).

**Proof.** We have \( G/N \times N \simeq G \) and by Corollary 1, \( k[G/N] \) is generated by \( p \)-polynomials in a fixed set of Frattini coordinates for \( G \). Say \( k[G/N] = k[l_1, \ldots, l_d] \subseteq k[G] \) where \( d = \text{codim } N \) and the \( l_i, i = 1, \ldots, d \), are \( p \)-polynomials in \( x_1, \ldots, x_n \). Then each \( l_i \) is constant on the fibres of \( n: G \rightarrow G/N \) and vanishes on \( N \).

Since \( k[G/N] \rightarrow k[G/N] \otimes k[N] = k[G] \) is a polynomial extension by [4,
Corollary 1 of Theorem 1] the ideal \((f_1, \cdots, f_d)k[G]\) is prime in \(k[G]\). Hence \(I(N) = (f_1, \cdots, f_d)k[G]\).

**Remarks.** 1. Corollary 2 is false without the assumption of normality on \(N \subset G\). Consider the following example suggested by Rosenlicht.

Let \(G\) be the group of \(3 \times 3\) upper triangular unipotent matrices

\[
G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in K, \ \text{char} \ K \neq 2 \right\}.
\]

Let \(x, y\) and \(z\) be the obvious Frattini coordinates. Then

\[ N = \left\{ \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : t \in K \right\} \]

is a connected subgroup of \(G\). The ideal \(I(N) = (x - y, z - x^2/2)\) is clearly not generated by two \(p\)-polynomials.

Moreover if \(H \subset G\) is the subgroup defined by \(x - y = 0\) and \(N \subset H\) is defined by \(x^p - x = z - x^2/2 = 0\), then \(N\) is a finite normal subgroup of \(H\) which cannot be defined by two \(p\)-polynomials in the Frattini coordinates \(x, z\) of \(H\). Thus the assumption of connectivity is also necessary in Corollary 2.

2. If \(G\) and \(k\) are as in Theorem 1 and \(H\) is any \(k\)-closed subgroup of codimension one (connected or not), then \(H\) can be defined by a single \(p\)-polynomial in any set of Frattini coordinates. More generally, any \(k\)-closed subgroup of \(G\) containing the Frattini subgroup of \(G\) can be defined by \(p\)-polynomials. Simply note that \(\text{codim}_G N = \text{codim}_{G/Fr(G)} N/Fr(G)\) and apply Proposition 2(i) and (ii).

3. An interesting application of Frattini coordinates is the following theorem of Sullivan.

**Theorem [6, Theorem 4].** A connected unipotent algebraic group defined over a field of characteristic \(p > 0\) is conservative if and only if it has dimension one.

**Proof.** Recall that an algebraic group is conservative if the following condition holds.

Let \(W\) be the group of all algebraic group automorphisms of \(G\). If \(f \in K[G]\) then \(V_f = \{w_*(f) : w \in W\}\) is finite dimensional.

By [2, §1] this is equivalent to saying that \(W\) may be given the structure of an algebraic group in such a way that the natural map \(W \times G \to G\) is a morphism of varieties.

Now let \(G = 1\) and \(K[G] = K[x_1, \cdots, x_n]\) where \(x_i, i = 1, \cdots, n,\)
are Frattini coordinates for $G$. Then it is easily checked (cf. [4, Corollary 2, p. 101]) that the assignments

\[
x_i \to x_i, \quad i = 1, \ldots, n - 1,
\]
\[
x_n \to x_n + P(x_1), \quad P \text{ a } p\text{-polynomial in } x_1,
\]
give an automorphism of $G$. In particular $V_{x_n}$ is not finite dimensional. It is well known that $\text{Aut}_{\text{Alg group}}(G_a) = G_m$ the multiplicative group.

4. If $\text{char } K = 0$, then with respect to the isomorphism of $G$ with $A^n$ given by a fixed set of Frattini coordinates, every normal subgroup is a linear subvariety.

5. The converse of Corollary 2 is easily seen to be false. If $G$ is the group of Remark 1 above and $H$ is the subgroup $y = z = 0$, it is easily seen that $H$ is not normal in $G$.

REFERENCES

7. J. Tits, Lectures on algebraic groups, Lecture Notes, Yale University, New Haven, Conn., 1967.