MORITA CONTEXTS OF ENRICHED CATEGORIES

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ABSTRACT. Categories enriched over a closed category $V$ are considered. The theorems and proofs are nonadditive while specializing when $V$ is the category of abelian groups to yield different interpretations and proofs of old results. $V$-adjoint equivalences of certain $V$-functor categories are shown to correspond to generalized Morita equivalences between small $V$-categories. Morita contexts are given a simple description as certain cospans and are shown to support a 2-dimensional structure.

For a bicomplete closed category $V$ [6, p. 180] we show that our generalized $V$-Morita equivalences between small $V$-categories correspond to $V$-adjoint equivalences between the corresponding $V$-functor categories. $V$-Morita equivalences are defined as Morita contexts invertible with respect to horizontal composition. $V$-Morita contexts are a special kind of diagram $(C_1 \rightarrow C \leftarrow C_0)$ in the category of small $V$-categories with horizontal composition induced by pushouts. In the classical case when $V$ is the category of abelian groups and $C_0$ and $C_1$ are each additive categories with one object, our definition of Morita context is equivalent to that of Bass [1], [2]. Our point of view is to consider Morita contexts as arrows in a bicategory (à la Bénabou [3, pp. 3–6]) and to apply a morphism which takes a Morita context into a left adjoint. We use right Kan extensions [4] to express our basic constructions.

We use two special $V$-categories $G$ and $2$, each of which has $\{0, 1\}$ as its set of objects, such that $G(i, j) = 1$ the unit object of $V$ and $2(i, j)$ is the terminal object of $V$ for all $i, j$. (If $V$ is cartesian closed, $G = 2$.) The $V$-category $[I]$ is the one object category with hom object $I$.

A $V$-Morita context $M$ is defined to be a pair $(C, T: C \rightarrow 2)$ where $C$
is a small $\mathbf{V}$-category and $T$ is a $\mathbf{V}$-functor, and a map of Morita contexts $F: (\mathbf{C}, T) \to (\mathbf{C}', T')$ is a $\mathbf{V}$-functor $F: \mathbf{C} \to \mathbf{C}'$ such that $T'F = T$. In other words, a Morita context $M = (\mathbf{C}, T)$ is a cospan $(\mathbf{C}, \xrightarrow{d_0} \mathbf{C}_0, \xleftarrow{d_1} \mathbf{C}_1)$ in the category of small $\mathbf{V}$-categories $\mathbf{C}$ in which $d_0$ and $d_1$ are the inclusions of the corresponding fibers of $T$ and the set of objects of $\mathbf{C}$ is the disjoint union of the sets of objects of $\mathbf{C}_0$ and $\mathbf{C}_1$, and a map of Morita contexts is a map of cospans. Morita contexts and their maps form a category $\mathcal{M}$ equipped with four important endofunctors. For $M = (\mathbf{C}, T)$, define the transpose of $M$ by $M' = (\mathbf{C}, rT)$ where $r: 2 \to 2$ is given by $r(i) = j$, $i \neq j$; define the opposite of $M$ by $M^o = (\mathbf{C}^o, T^o)$. For a Morita context $M = (\mathbf{C}_1 \hookrightarrow \mathbf{C} \leftarrow \mathbf{C}_0)$ define the left identity of $M$ by $\lambda(M) = (\mathbf{C}_1 \otimes \mathbf{C}_0, L)$ where $L$ is induced by projection on $\mathbf{C}_0$; define the right identity of $M$ by $\rho(M) = \lambda(M')$.

If $M = (\mathbf{C}_1 \hookrightarrow \mathbf{C} \leftarrow \mathbf{C}_0)$ and $M' = (\mathbf{C}_1' \hookrightarrow \mathbf{C}' \leftarrow \mathbf{C}_0')$ are Morita contexts with $\lambda(M) = \lambda(M')$, define the $*$-composite $M * M'$ by first taking the composite of the cospans, i.e., let

$\begin{array}{c}
\mathbf{C}_1 \\
\mathbf{C}
\end{array} \quad \begin{array}{c}
\mathbf{C}_0
\mathbf{C}
\mathbf{C}_1
\end{array} \quad \begin{array}{c}
\mathbf{C}_0
\mathbf{C}
\mathbf{C}_1
\end{array}
$ be a pushout in the category of small $\mathbf{V}$-categories and obtain the cospan

$(\mathbf{C}_1 \downarrow \mathbf{C}_0 \mathbf{C}' \leftarrow \mathbf{C}_1')$, and then let the category of $M * M'$ be the full subcategory of $\mathbf{C} \downarrow \mathbf{C}_0 \mathbf{C}'$ with objects the disjoint union of the objects of $\mathbf{C}_1$ and $\mathbf{C}_1'$. We note that for $X \in |\mathbf{C}_1|'$ and $Z \in |\mathbf{C}_1|$, $(\mathbf{C} \downarrow \mathbf{C}_0 \mathbf{C}')(X, Z)$ is the coend [4] over all $Y \in |\mathbf{C}_1|$ of $\mathbf{C}(Y, Z) \otimes \mathbf{C}'(X, Y)$. If $\phi: M \to N$ and $\phi': M' \to N'$ are maps of Morita contexts such that $\lambda(\phi) = \lambda(\phi')$ then there is a map $\phi * \phi': M * M' \to N * N'$ by the universal property of pushouts. This $*$-composition is associative up to isomorphism since the composition of cospans is. There are left and right identity isomorphisms $l_M: \lambda(M) * M \to M$ and $r_M: M * \rho(M) \to M$.

Theorem 1. If for small $\mathbf{V}$-categories $\mathbf{C}_0$ and $\mathbf{C}_1$ we define the category $\mathcal{B}(\mathbf{C}_0, \mathbf{C}_1)$ to have as objects Morita contexts $(\mathbf{C}_1 \hookrightarrow \mathbf{C} \leftarrow \mathbf{C}_0)$ and to have as maps only the maps of Morita contexts which are the identity on the fibers, then $*$-composition becomes a functor

$\ast: \mathcal{B}(\mathbf{C}_1, \mathbf{C}_2) \times \mathcal{B}(\mathbf{C}_0, \mathbf{C}_1) \to \mathcal{B}(\mathbf{C}_0, \mathbf{C}_2)$

and $\mathcal{B}$ is a bicategory in the sense of Bénabou [3, pp. 3-6]. $\square$

Define maps $\lambda_M: M * M^o \to \lambda(M)$ and $\rho_M: M^o * M \to \rho(M)$ such that $\lambda_M$
is the identity on the fibers and $\lambda_M$ on the other hom objects is induced by the compositions
\[
\{ C(Y, Z) \otimes C(X, Y) \to C(X, Z) | Y \in C_0 \}
\]
and $\rho_M = \lambda_{M^t}$. Ignoring associativity and left and right identity isomorphisms we have equations:

1. \[(\lambda_M * M) = (M * M^t) * M \to M;\]

2. \[\lambda_{M^* M^t} = \lambda_M \cdot (M * \lambda_{M^t} * M^t): (M * M^t) * (M * M^t)^t \to I(M);\]

3. \[\lambda_N \cdot (\phi * \phi^t) = I(\phi) \cdot \lambda_M: M * M^t \to I(N);\]

for Morita contexts $M$, $M'$, and $N$ with $\lambda(M) = I(M')$ and $\phi: M \to N$ a map of Morita contexts.

If $V$ is the category of abelian groups, the correspondence between our Morita contexts and those of Newell [7] which are 4-tuples $(U, V, \mu, \nu)$ is given as follows: $M = (C_1 \to C \to C_0)$ corresponds to the 4-tuple with $U$ (respectively, $V$) the restriction of the enriched hom of $C$ to $C_0 \otimes C_1$ (respectively, $C_1 \otimes C_0$) and $\mu$ and $\nu$ the transformations induced by $\rho_M$ and $\lambda_M$, respectively.

**Theorem 2.** The following statements are equivalent for a Morita context $M$.

(a) There exists a Morita context $M'$ such that $M * M' \cong I(M)$, i.e., $M$ has a right $*$-inverse.

(b) $\lambda_M: M * M^t \to I(M)$ is a split epimorphism.

(c) $\lambda_M$ is an isomorphism.

(d) $M^t$ has a left $*$-inverse.

(e) $M^0$ has a right $*$-inverse.

**Proof.** The only hard part is (a) $\Rightarrow$ (b) $\Rightarrow$ (c). To show (a) $\Rightarrow$ (b) we note that if $\phi: M * M' \to I(M)$ is the isomorphism then equations (2) and (3) above give
\[
\lambda_{I(M)} \cdot (\phi * \phi^t) = I(\phi) \cdot \lambda_M \cdot (M * \lambda_{M^t} * M^t).
\]
Since $\lambda_{I(M)}$ is an isomorphism and so are $I(\phi)$ and $\phi * \phi^t$, we have $\lambda_M$ is a split epimorphism.

If (b) holds then there is an $s: I(M) \to M * M^t$ such that $\lambda_M s = I(M)$. Equation (1) applied to both $M$ and $M^t$ gives us that $s\lambda_M = (\lambda_M s) * M * M^t$ if we ignore all associativity and left and right identity isomorphisms. Hence $s\lambda_M = I(M)$.

\[\square\]
A Morita context is said to be a Morita equivalence if it has both a left and a right \(*\)-inverse. Examples are \(\mathcal{L}(M)\) and \(\mathcal{L}(M)\).

There is another binary operation \(\Box\) on Morita contexts (which we might call vertical composition) which is always defined and is associative and commutative up to isomorphism. Namely, if \(M = (C_1 \to C \to C_0)\) and \(N = (D_1 \to D \to D_0)\) are Morita contexts \(M \Box N\) has as its category the full subcategory of \(C \otimes D\) with fibers \(C_1 \otimes D_1\) and \(C_0 \otimes D_0\). For Morita contexts \(M, M', N\) and \(N'\) such that \(\mathcal{L}(M) = \mathcal{L}(M')\) and \(\mathcal{L}(N) = \mathcal{L}(N')\) we have equations:

\[(M \Box N \Box N') = (M \Box M') \Box (N \Box N');\]

\[(M \Box N)^t = M^t \Box N^t \quad \text{and} \quad \rho_{M \Box N} = \rho_M \Box \rho_N.\]

**Theorem 3.** Let \(M\) be a Morita context.

(i) If \(M\) is a Morita equivalence so are \(M^t\) and \(M^0\).

(ii) If \(M\) and \(M'\) are Morita equivalences with \(\mathcal{L}(M) = \mathcal{L}(M')\), then \(M \Box M'\) is a Morita equivalence.

(iii) If \(M\) and \(N\) have left \(*\)-inverses so does \(M \Box N\).

(iv) If \(\mathcal{L}(M) = \mathcal{L}(N) = ([\mathbb{1}] \to [\mathbb{1}] \otimes G \leftarrow [\mathbb{1}])\) and \(M \Box N\) is a Morita equivalence, then \(M\) and \(N\) are Morita equivalences.

**Proof.** Statements (i), (ii) and (iii) follow from Theorem 2 and equations (2) and (5). If the hypotheses of (iv) hold then equation (4) and the equalities \(M = M \Box \mathcal{L}(N)\) and \(N = \mathcal{L}(M) \Box N\) yield the equations \(M \Box N = (\mathcal{L}(M) \Box N) \Box M\) and \(M \Box N = (M \Box \mathcal{L}(N)) \Box N\), from which the conclusions of (iv) follow. \(\Box\)

Let \(\text{Lad}\) be the 2-dimensional category with objects small \(\mathcal{V}\)-categories \(C\) and \(\text{Lad}(C, C')\) the category of \(\mathcal{V}\)-functors from \(\mathcal{V}C\) to \(\mathcal{V}C'\) which are \(\mathcal{V}\)-left adjoints, i.e., \(\mathcal{V}\)-cocontinuous, with maps \(\mathcal{V}\)-natural transformations.

There is a strict homomorphism of bicategories \(\Phi: B \to \text{Lad}\) defined by

\[
\Phi(C) = C \quad \text{and} \quad \Phi(C_1 \xrightarrow{d_1} C \xrightarrow{d_0} C_0) = \text{Ran}_{d_0}^{d_1} V^{d_1} = V^{d_1} \cdot (V^{d_0})^t,
\]

where we have computed the right Kan extension in terms of \((V^{d_0})^t\), the left adjoint of \(V^{d_0}\). Note that

\[
\Phi(M)(C_0(X, -))(Y) = C(X, Y)
\]

for \(X\) in \(C_0\) and \(Y\) in \(C_1\). We then have natural transformations

\[
\Phi(\lambda_M): \Phi(M) \cdot \Phi(M^t) \to \Phi(\mathcal{L}(M)) = \text{id}_{\mathcal{V}C_1}
\]

and

\[
\Phi(\rho_M): \Phi(M^t) \cdot \Phi(M) \to \Phi(\mathcal{L}(M)) = \text{id}_{\mathcal{V}C_0}
\]
and for $X$ and $Z$ in $C_0$

$\Phi(\rho_M)(C_0(X, -))(Z) = \rho_M(X, Z).$

Theorem 4. If

\[ M = \left( \begin{array}{ccc} C_1 & d_1 & C_0 \\ C_0 & d_0 & C_1 \end{array} \right) \]

is a Morita context with a right $*$-inverse, then the following are true:

(i) $\Phi(M^t)$ is left adjoint to $\Phi(M)$ with counit $\Phi(\rho_M)$ and unit $\Phi(\lambda_M)^{-1}$ which is an isomorphism.

(ii) The functor $\Phi(M^t)$ maps $V$-atoms [5, (4.3)] into $V$-atoms and hence representables into $V$-atoms.

(iii) The functor

\[ C_0 \xrightarrow{R} V C \xrightarrow{d_0} V C_0, \]

where $R$ is the Yoneda embedding, is $V$-full and faithful, i.e., $d_0: C_0 \to C$ is $V$-codense [4].

(iv) $M$ has a left $*$-inverse if and only if

\[ C_1 \xrightarrow{R_1} V C_1 \Phi(M^t) \xrightarrow{\Phi(M^t)} V C_0 \]

is $V$-dense.

Proof. (i) is a consequence of applying $\Phi$ which is a strict map of bicategories to equation (1) for $M$ and $M^t$. (ii) follows from the fact that $\Phi(M)$ is the right adjoint of $\Phi(M^t)$ and is $V$-cocontinuous. Thus for $G$ a $V$-atom in $V^{C_1}$, we have

\[ V^{C_0}(\Phi(M^t)G, -) \cong V^{C_1}(G, -) \cdot \Phi(M) \]

which is $V$-cocontinuous. Part (iii) is equivalent to stating both that

\[ C_0 \xrightarrow{R_1} V C_1 \Phi(M^t) \xrightarrow{\Phi(M^t)} V C_0 \]

is $V$-full and faithful, which is true since $R_1$ and $\Phi(M^t)$ are, and that

\[ \Phi(M)(C_0(C_0, -))C_1 = V^{C_0}(\Phi(M^t)R_1(C_1), C_0(C_0, -)) \]

for $C_0$ in $C_0$ and $C_1$ in $C_1$, which holds since $\Phi(M)$ is $V$-right adjoint to $\Phi(M^t)$.

To show (iv) we note that since $\Phi(M^t)$ is $V$-cocontinuous, $\Phi(M^t) \cdot R_1$ is $V$-dense if and only if $\Phi(M^t)$ is $V$-dense. But $\Phi(M^t)$ is $V$-dense if and
only if its right adjoint is full and faithful, i.e., $\Phi(p_M)$ is an isomorphism. But by (6) $\Phi(p_M)$ is an isomorphism if and only if $p_M$ is one. □

**Corollary 5.** The map of bicategories $\Phi: B \rightarrow \text{Lad}$ induces an isomorphism of the Picard groupoids [3, p. 57]

$$\hat{\Phi}: \text{Pic } B \rightarrow \text{Pic } (\text{Lad}) \quad □$$

$(\text{Pic } B)(C_0, C_1)$ is the set of isomorphism classes of Morita equivalences, i.e., invertible arrows from $C_0$ to $C_1$.

**REFERENCES**

2. ———, *The Morita theorems*, University of Oregon (mimeographed notes).

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