MORITA CONTEXTS OF ENRICHED CATEGORIES

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ABSTRACT. Categories enriched over a closed category $V$ are considered. The theorems and proofs are nonadditive while specializing when $V$ is the category of abelian groups to yield different interpretations and proofs of old results. $V$-adjoint equivalences of certain $V$-functor categories are shown to correspond to generalized Morita equivalences between small $V$-categories. Morita contexts are given a simple description as certain cospans and are shown to support a 2-dimensional structure.

For a bicomplete closed category $V$ [6, p. 180] we show that our generalized $V$-Morita equivalences between small $V$-categories correspond to $V$-adjoint equivalences between the corresponding $V$-functor categories. $V$-Morita equivalences are defined as Morita contexts invertible with respect to horizontal composition. $V$-Morita contexts are a special kind of diagram $(C_1 \rightarrow C \leftarrow C_0)$ in the category of small $V$-categories with horizontal composition induced by pushouts. In the classical case when $V$ is the category of abelian groups and $C_0$ and $C_1$ are each additive categories with one object, our definition of Morita context is equivalent to that of Bass [1], [2]. Our point of view is to consider Morita contexts as arrows in a bicategory (à la Bénabou [3, pp. 3–6]) and to apply a morphism which takes a Morita context into a left adjoint. We use right Kan extensions [4] to express our basic constructions.

We use two special $V$-categories $G$ and $2$, each of which has $\{0, 1\}$ as its set of objects, such that $G(i, j) = 1$ the unit object of $V$ and $2(i, j)$ is the terminal object of $V$ for all $i, j$. (If $V$ is cartesian closed, $G = 2$.) The $V$-category $[I]$ is the one object category with hom object $1$.

A $V$-Morita context $M$ is defined to be a pair $(C, T: C \rightarrow 2)$ where $C$...
is a small $V$-category and $T$ is a $V$-functor, and a map of Morita contexts $F: (C, T) \to (C', T')$ is a $V$-functor $F: C \to C'$ such that $T'F = T$. In other words, a Morita context $M = (C, T)$ is a cospan $(C \overset{j_1}{\longrightarrow} C' \overset{j}{\longrightarrow} C_0)$ in the category of small $V$-categories \cite{2} in which $j_0$ and $j_1$ are the inclusions of the corresponding fibers of $T$ and the set of objects of $C$ is the disjoint union of the sets of objects of $C_0$ and $C_1$, and a map of Morita contexts is a map of cospans. Morita contexts and their maps form a category $M$ equipped with four important endofunctors. For $M = (C, T)$, define the transpose of $M$ by $M' = (C, rT)$ where $r: 2 \to 2$ is given by $r(i) = j$, $i \neq j$; define the opposite of $M$ by $M^\circ = (C^0, T^0)$. For a Morita context $M = (C_1 \hookrightarrow C \hookleftarrow C_0)$ define the left identity of $M$ by $\lambda(M) = (C_1 \boxtimes G, L)$ where $L$ is induced by projection on $G$; define the right identity of $M$ by $\rho(M)$.

If $M = (C_1 \hookrightarrow C \hookleftarrow C_0)$ and $M' = (C_1' \hookrightarrow C' \hookleftarrow C'_0)$ are Morita contexts with $\lambda(M) = \lambda(M')$, define the $\ast$-composite $M \ast M'$ by first taking the composite of the cospans, i.e., let

\[
\begin{array}{ccc}
C_0 & \longrightarrow & C' \\
\downarrow & & \downarrow \\
C & \longrightarrow & C' \\
\end{array}
\]

be a pushout in the category of small $V$-categories and obtain the cospan $(C_1 \sqcup_{C_0} C' \sqcup_{C_0} C')$, and then let the category of $M \ast M'$ be the full subcategory of $C \sqcup_{C_0} C'$ with objects the disjoint union of the objects of $C_1$ and $C'_0$. We note that for $X \in |C_0|$ and $Z \in |C_1|$, $(C \sqcup_{C_0} C')(X, Z)$ is the coend \cite{4} over all $Y \in |C_0|$ of $C(Y, Z) \boxtimes C'(X, Y)$. If $\phi: M \to N$ and $\phi': M' \to N'$ are maps of Morita contexts such that $\lambda(\phi) = \lambda(\phi')$ then there is a map $\phi \ast \phi': M \ast M' \to N \ast N'$ by the universal property of pushouts. This $\ast$-composition is associative up to isomorphism since the composition of cospans is. There are left and right identity isomorphisms $l_M: \lambda(M) \ast M \to M$ and $r_M: M \ast \rho(M) \to M$.

**Theorem 1.** If for small $V$-categories $C_0$ and $C_1$ we define the category $B(C_0, C_1)$ to have as objects Morita contexts $(C_1 \hookrightarrow C \hookleftarrow C_0)$ and to have as maps only the maps of Morita contexts which are the identity on the fibers, then $\ast$-composition becomes a functor

$$ \ast: B(C_1, C_2) \times B(C_0, C_1) \to B(C_0, C_2) $$

and $B$ is a bicategory in the sense of Bénabou \cite[pp. 3–6]{3}. \hfill \Box

Define maps $\lambda_M: M \ast M^t \to \lambda(M)$ and $\rho_M: M^t \ast M \to \rho(M)$ such that $\lambda_M$
is the identity on the fibers and $\lambda_M$ on the other hom objects is induced by the compositions

$$\{C(Y, Z) \otimes C(X, Y) \to C(X, Z) \mid Y \in C_0\}$$

and $\rho_M = \lambda_M^t$. Ignoring associativity and left and right identity isomorphisms we have equations:

1. \((\lambda_M \ast M) = (M \ast \rho_M) : M \ast M^t \ast M \to M\);
2. \(\lambda_M \ast M^t = \lambda_M \cdot (M \ast \lambda_M^t \ast M^t) : (M \ast M^t) \ast (M \ast M^t)^t \to I(M)\);
3. \(\lambda^N \ast (\phi \ast \phi^t) = l(\phi) \cdot \lambda_M^t : M \ast M^t \to I(N)\);

for Morita contexts $M, M', \text{and } N$ with $\lambda(M) = I(M')$ and $\phi : M \to N$ a map of Morita contexts.

If $V$ is the category of abelian groups, the correspondence between our Morita contexts and those of Newell [7] which are 4-tuples $(U, V, \mu, \nu)$ is given as follows: $M = (C_1 \to C \to C_0)$ corresponds to the 4-tuple with $U$ (respectively, $V$) the restriction of the enriched hom of $C$ to $C_0 \otimes C_1$ (respectively, $C_1 \otimes C_0$) and $\mu$ and $\nu$ the transformations induced by $\rho_M$ and $\lambda_M$, respectively.

**Theorem 2.** The following statements are equivalent for a Morita context $M$.

(a) There exists a Morita context $M'$ such that $M \ast M' \cong I(M)$, i.e., $M$ has a right $\ast$-inverse.
(b) $\lambda^M : M \ast M^t \to I(M)$ is a split epimorphism.
(c) $\lambda^M$ is an isomorphism.
(d) $M^t$ has a left $\ast$-inverse.
(e) $M^t$ has a right $\ast$-inverse.

**Proof.** The only hard part is (a) $\Rightarrow$ (b) $\Rightarrow$ (c). To show (a) $\Rightarrow$ (b) we note that if $\phi : M \ast M' \to I(M)$ is the isomorphism then equations (2) and (3) above give

$$\lambda_{I(M)} \cdot (\phi \ast \phi^t) = I(\phi) \cdot \lambda^M \cdot (M \ast \lambda_M^t, \ast M^t).$$

Since $\lambda_{I(M)}$ is an isomorphism and so are $I(\phi)$ and $\phi \ast \phi^t$, we have $\lambda_M$ is a split epimorphism.

If (b) holds then there is an $s : I(M) \to M \ast M'$ such that $\lambda_M^t s = I(M)$. Equation (1) applied to both $M$ and $M'$ gives us that $s\lambda^M = (\lambda_M^t s) \ast M \ast M'$ if we ignore all associativity and left and right identity isomorphisms. Hence $s\lambda^M = I(M)$.
A Morita context is said to be a Morita equivalence if it has both a left and a right \(*\)-inverse. Examples are \(I(M)\) and \(\mathcal{A}(M)\).

There is another binary operation \(\square\) on Morita contexts (which we might call vertical composition) which is always defined and is associative and commutative up to isomorphism. Namely, if \(M = (C_1 \rightarrow C \leftarrow C_0)\) and \(N = (D_1 \rightarrow D \leftarrow D_0)\) are Morita contexts \(M \square N\) has as its category the full subcategory of \(C \otimes D\) with fibers \(C_1 \otimes D_1\) and \(C_0 \otimes D_0\). For Morita contexts \(M, M', N\) and \(N'\) such that \(\mathcal{A}(M) = I(M')\) and \(\mathcal{A}(N) = I(N')\) we have equations:

\[
(4) \quad (M \square N) \ast (M' \square N') = (M \ast M') \square (N \ast N');
\]
\[
(5) \quad (M \square N)^t = M^t \square N^t \quad \text{and} \quad \rho_{M \square N} = \rho_M \square \rho_N.
\]

**Theorem 3.** Let \(M\) be a Morita context.

(i) If \(M\) is a Morita equivalence so are \(M^t\) and \(M^0\).

(ii) If \(M\) and \(M'\) are Morita equivalences with \(\mathcal{A}(M) = I(M')\), then \(M \ast M'\) is a Morita equivalence.

(iii) If \(M\) and \(N\) have left \(*\)-inverses so does \(M \square N\).

(iv) If \(\mathcal{A}(M) = \mathcal{A}(N) = ([I] \rightarrow [I] \otimes G \leftarrow [I])\) and \(M \square N\) is a Morita equivalence, then \(M\) and \(N\) are Morita equivalences.

**Proof.** Statements (i), (ii) and (iii) follow from Theorem 2 and equations (2) and (5). If the hypotheses of (iv) hold then equation (4) and the equalities \(M = M \square \mathcal{A}(N)\) and \(N = \mathcal{A}(M) \square N\) yield the equations \(M \square N = (I(M) \square N) \ast M\) and \(M \square N = (M \square I(N)) \ast N\), from which the conclusions of (iv) follow. □

Let \(\text{Lad}\) be the 2-dimensional category with objects small \(\mathcal{V}\)-categories \(C\) and \(\text{Lad}(C, C')\) the category of \(\mathcal{V}\)-functors from \(\mathcal{V}C\) to \(\mathcal{V}C'\) which are \(\mathcal{V}\)-left adjoints, i.e., \(\mathcal{V}\)-cocontinuous, with maps \(\mathcal{V}\)-natural transformations. There is a strict homomorphism of bicategories \(\Phi: \text{B} \rightarrow \text{Lad}\) defined by

\[
\Phi(C) = C \quad \text{and} \quad \Phi \left( \begin{array}{c} C_1 \rightarrow C \rightarrow C_0 \\ d_1 \quad d_0 \end{array} \right) = \text{Ran}_{d_0} v^{d_1} = v^{d_1} \cdot (v^{d_0})^t,
\]

where we have computed the right Kan extension in terms of \((v^{d_0})^t\), the left adjoint of \(v^{d_0}\). Note that

\[
\Phi(M)(C_0(X, -))(Y) = C(X, Y)
\]

for \(X\) in \(C_0\) and \(Y\) in \(C_1\). We then have natural transformations \(\Phi(\lambda_M): \Phi(M) \cdot \Phi(M^t) \rightarrow \Phi(I(M)) = \text{id}_{\mathcal{V}C_1}\) and

\[
\Phi(\rho_M): \Phi(M^t) \cdot \Phi(M) \rightarrow \Phi(\mathcal{A}(M)) = \text{id}_{\mathcal{V}C_0}.
\]
and for $X$ and $Z$ in $C_0$

$\Phi(p_M)(C_0(X, -))(Z) = p_M(X, Z)$.  

Theorem 4.  If

$$M = \left( \begin{array}{ccc} d_1 & d_0 \\ \downarrow & \downarrow \\ C_1 & C_0 \end{array} \right)$$

is a Morita context with a right $\ast$-inverse, then the following are true:

(i) $\Phi(M^t)$ is left adjoint to $\Phi(M)$ with counit $\Phi(p_M)$ and unit $\Phi(\lambda_M)^{-1}$ which is an isomorphism.

(ii) The functor $\Phi(M^t)$ maps $V$-atoms [5, (4.3)] into $V$-atoms and hence representables into $V$-atoms.

(iii) The functor

$$C^o \xrightarrow{R} V \xrightarrow{d_0} V^C,$$

where $R$ is the Yoneda embedding, is $V$-full and faithful, i.e., $d_0: C_0 \to C$ is $V$-codense [4].

(iv) $M$ has a left $\ast$-inverse if and only if

$$C^o \xrightarrow{R_1} V^C \xrightarrow{\Phi(M^t)} V^{C_0}$$

is $V$-dense.

Proof.  (i) is a consequence of applying $\Phi$ which is a strict map of bicategories to equation (1) for $M$ and $M^t$.  (ii) follows from the fact that $\Phi(M)$ is the right adjoint of $\Phi(M^t)$ and is $V$-cocontinuous.  Thus for $G$ a $V$-atom in $V^C$, we have

$$V^{C_0}(\Phi(M^t)G, -) \simeq V^C1(G, -) \cdot \Phi(M)$$

which is $V$-cocontinuous.  Part (iii) is equivalent to stating both that

$$C^o \xrightarrow{R_1} V^C \xrightarrow{\Phi(M^t)} V^{C_0}$$

is $V$-full and faithful, which is true since $R_1$ and $\Phi(M^t)$ are, and that

$$\Phi(M)(C_0(C_0, -))C_1 = V^{C_0}(\Phi(M^t)R_1(C_1), C_0(C_0, -))$$

for $C_0$ in $C_0$ and $C_1$ in $C_1$, which holds since $\Phi(M)$ is $V$-right adjoint to $\Phi(M^t)$.

To show (iv) we note that since $\Phi(M^t)$ is $V$-cocontinuous, $\Phi(M^t) \cdot R_1$ is $V$-dense if and only if $\Phi(M^t)$ is $V$-dense.  But $\Phi(M^t)$ is $V$-dense if and
only if its right adjoint is full and faithful, i.e., $\Phi(p_M)$ is an isomorphism.
But by (6) $\Phi(p_M)$ is an isomorphism if and only if $p_M$ is one.  \hfill \Box

**Corollary 5.** The map of bicategories $\Phi: B \to \text{Lad}$ induces an isomorphism of the Picard groupoids [3, p. 57]

\[ \hat{\Phi}: \text{Pic } B \to \text{Pic } (\text{Lad}). \]  \hfill \Box

$(\text{Pic } B)(C_0, C_1)$ is the set of isomorphism classes of Morita equivalences, i.e., invertible arrows from $C_0$ to $C_1$.

**REFERENCES**

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