MORITA CONTEXTS OF ENRICHED CATEGORIES

J. FISHER-PALMQUIST AND P. H. PALMQUIST

ABSTRACT. Categories enriched over a closed category \( V \) are considered. The theorems and proofs are nonadditive while specializing when \( V \) is the category of abelian groups to yield different interpretations and proofs of old results. \( V \)-adjoint equivalences of certain \( V \)-functor categories are shown to correspond to generalized Morita equivalences between small \( V \)-categories. Morita contexts are given a simple description as certain cospans and are shown to support a 2-dimensional structure.

For a bicomplete closed category \( V \) \([6, p. 180]\) we show that our generalized \( V \)-Morita equivalences between small \( V \)-categories correspond to \( V \)-adjoint equivalences between the corresponding \( V \)-functor categories. \( V \)-Morita equivalences are defined as Morita contexts invertible with respect to horizontal composition. \( V \)-Morita contexts are a special kind of diagram \( (C_1 \rightarrow C \leftarrow C_0) \) in the category of small \( V \)-categories with horizontal composition induced by pushouts. In the classical case when \( V \) is the category of abelian groups and \( C_0 \) and \( C_1 \) are each additive categories with one object, our definition of Morita context is equivalent to that of Bass \([1], [2]\).

Our point of view is to consider Morita contexts as arrows in a bicategory (à la Bénabou \([3, pp. 3–6]\)) and to apply a morphism which takes a Morita context into a left adjoint. We use right Kan extensions \([4]\) to express our basic constructions.

We use two special \( V \)-categories \( G \) and \( 2 \), each of which has \( \{0, 1\} \) as its set of objects, such that \( G(i, j) = 1 \) the unit object of \( V \) and \( 2(i, j) \) is the terminal object of \( V \) for all \( i, j \). (If \( V \) is cartesian closed, \( G = 2 \).

The \( V \)-category \([I]\) is the one object category with hom object \( I \).

A \( V \)-Morita context \( M \) is defined to be a pair \((C, T : C \rightarrow 2)\) where \( C \)
is a small $\mathbf{V}$-category and $T$ is a $\mathbf{V}$-functor, and a map of Morita contexts $F: (C, T) \to (C', T')$ is a $\mathbf{V}$-functor $F: C \to C'$ such that $T'F = T$. In other words, a Morita context $M = (C, T)$ is a cospan $(C \leftarrow C_1 \rightarrow C_0)$ in the category of small $\mathbf{V}$-categories [2] in which $d_0$ and $d_1$ are the inclusions of the corresponding fibers of $T$ and the set of objects of $C$ is the disjoint union of the sets of objects of $C_0$ and $C_1$, and a map of Morita contexts is a map of cospans. Morita contexts and their maps form a category $\mathcal{M}$ equipped with four important endofunctors. For $M = (C, T)$, define the transpose of $M$ by $M^t = (C, rT)$ where $r: 2 \to 2$ is given by $r(i) = j$, $i \neq j$; define the opposite of $M$ by $M^o = (C^o, T^o)$. For a Morita context $M = (C_1 \leftarrow C \rightarrow C_0)$ define the left identity of $M$ by $l(M) = (C_1 \otimes G, L)$ where $L$ is induced by projection on $G$; define the right identity of $M$ by $r(M) = l(M^t)$.

If $M = (C_1 \leftarrow C \rightarrow C_0)$ and $M' = (C_1' \leftarrow C' \rightarrow C_0')$ are Morita contexts with $l(M) = l(M')$, define the **-composite $M \star M'$ by first taking the composite of the cospans, i.e., let

$$
\begin{array}{ccc}
C_0 & \longrightarrow & C' \\
\downarrow & & \downarrow \\
C & \longrightarrow & C' \\
\end{array}
$$

be a pushout in the category of small $\mathbf{V}$-categories and obtain the cospan $(C_1 \leftarrow C \downarrow C_0 \rightarrow C' \leftarrow C_0')$, and then let the category of $M \star M'$ be the full subcategory of $C \downarrow C_0 \downarrow C'$ with objects the disjoint union of the objects of $C_1$ and $C_0'$. We note that for $X \in |C_0'|$ and $Z \in |C_0|$, $(C \downarrow C_0 \downarrow C')(X, Z)$ is the coend [4] over all $Y \in |C_0|$ of $C(Y, Z) \otimes C'(X, Y)$. If $\phi: M \to N$ and $\phi': M' \to N'$ are maps of Morita contexts such that $r(\phi) = l(\phi')$ then there is a map $\phi \star \phi': M \star M' \to N \star N'$ by the universal property of pushouts. This **-composition is associative up to isomorphism since the composition of cospans is. There are left and right identity isomorphisms $l_M: l(M) \star M \to M$ and $r_M: M \star r(M) \to M$.

**Theorem 1.** If for small $\mathbf{V}$-categories $C_0$ and $C_1$ we define the category $B(C_0, C_1)$ to have as objects Morita contexts $(C_1 \leftarrow C \rightarrow C_0)$ and to have as maps only the maps of Morita contexts which are the identity on the fibers, then **-composition becomes a functor

$$**: B(C_0, C_1) \times B(C_0, C_1) \to B(C_0, C_1)$$

and $B$ is a bicategory in the sense of Bénabou [3, pp. 3–6]. □

Define maps $\lambda_M: M \star M^t \to l(M)$ and $\rho_M: M^t \star M \to r(M)$ such that $\lambda_M$
is the identity on the fibers and \( \lambda_M \) on the other hom objects is induced by the compositions
\[
\{ C(Y, Z) \otimes C(X, Y) \to C(X, Z) \mid Y \in C_0 \}
\]
and \( \rho_M = \lambda_{M^t} \). Ignoring associativity and left and right identity isomorphisms we have equations:

1. \( (\lambda_M * M) = (M * \rho_M): M * M^t * M \to M; \)
2. \( \lambda_{M * M^t} = \lambda_M \cdot (M * \lambda_{M^t} * M^t): (M * M^t) * (M * M^t)^t \to \mathcal{I}(M); \)
3. \( \lambda_N \cdot (\phi * \phi^t) = \mathcal{I}(\phi) \cdot \lambda_M: M * M^t \to \mathcal{I}(N); \)

for Morita contexts \( M, M', \) and \( N \) with \( \mathcal{I}(M) = \mathcal{I}(M') \) and \( \phi: M \to N \) a map of Morita contexts.

If \( V \) is the category of abelian groups, the correspondence between our Morita contexts and those of Newell [7] which are 4-tuples \( (U, V, \mu, \nu) \) is given as follows: \( M = (C_1 \to C \to C_0) \) corresponds to the 4-tuple with \( U \) (respectively, \( V \)) the restriction of the enriched hom of \( C \) to \( C_0 \otimes C_1 \) (respectively, \( C_1 \otimes C_0 \)) and \( \mu \) and \( \nu \) the transformations induced by \( \rho_M \) and \( \lambda_M \), respectively.

**Theorem 2.** The following statements are equivalent for a Morita context \( M \).

- (a) There exists a Morita context \( M' \) such that \( M * M' \cong \mathcal{I}(M) \), i.e., \( M \) has a right \(*\)-inverse.
- (b) \( \lambda_M: M * M^t \to \mathcal{I}(M) \) is a split epimorphism.
- (c) \( \lambda_M \) is an isomorphism.
- (d) \( M^t \) has a left \(*\)-inverse.
- (e) \( M^t \) has a right \(*\)-inverse.

**Proof.** The only hard part is (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c). To show (a) \( \Rightarrow \) (b) we note that if \( \phi: M * M^t \to \mathcal{I}(M) \) is the isomorphism then equations (2) and (3) above give
\[
\lambda_{\mathcal{I}(M)} \cdot (\phi * \phi^t) = \mathcal{I}(\phi) \cdot \lambda_M \cdot (M * \lambda_{M^t} * M^t).
\]
Since \( \lambda_{\mathcal{I}(M)} \) is an isomorphism and so are \( \mathcal{I}(\phi) \) and \( \phi * \phi^t \), we have \( \lambda_M \) is a split epimorphism.

If (b) holds then there is an \( s: \mathcal{I}(M) \to M * M^t \) such that \( \lambda_M s = \text{id}_{\mathcal{I}(M)} \).

Equation (1) applied to both \( M \) and \( M^t \) gives us that \( s\lambda_M = (\lambda_M s) * M * M^t \) if we ignore all associativity and left and right identity isomorphisms. Hence \( s\lambda_M = \text{id}_{M * M^t} \). \( \square \)
A Morita context is said to be a Morita equivalence if it has both a left and a right \(*\)-inverse. Examples are \(\mathcal{I}(M)\) and \(\mathcal{J}(M)\).

There is another binary operation \(\Box\) on Morita contexts (which we might call vertical composition) which is always defined and is associative and commutative up to isomorphism. Namely, if \(M = (C_1 \to C \to C_0)\) and \(N = (D_1 \to D \to D_0)\) are Morita contexts \(M \Box N\) has as its category the full subcategory of \(C \otimes D\) with fibers \(C_1 \otimes D_1\) and \(C_0 \otimes D_0\). For Morita contexts \(M, M', N\) and \(N'\) such that \(\mathcal{I}(M) = \mathcal{I}(M')\) and \(\mathcal{J}(N) = \mathcal{I}(N')\) we have equations:

\[
\begin{align*}
(4) \quad (M \Box N) \ast (M' \Box N') &= (M \ast M') \Box (N \ast N'); \\
(5) \quad (M \Box N)^t &= M^t \Box N^t \quad \text{and} \quad \rho_{M \Box N} = \rho_M \Box \rho_N.
\end{align*}
\]

**Theorem 3.** Let \(M\) be a Morita context.

(i) If \(M\) is a Morita equivalence so are \(M^t\) and \(M^0\).

(ii) If \(M\) and \(M'\) are Morita equivalences with \(\mathcal{I}(M) = \mathcal{I}(M')\), then \(M \ast M'\) is a Morita equivalence.

(iii) If \(M\) and \(N\) have left \(*\)-inverses so does \(M \Box N\).

(iv) If \(\mathcal{I}(M) = \mathcal{J}(N) = ([1] \to [1] \otimes G \leftarrow [1])\) and \(M \Box N\) is a Morita equivalence, then \(M\) and \(N\) are Morita equivalences.

**Proof.** Statements (i), (ii) and (iii) follow from Theorem 2 and equations (2) and (5). If the hypotheses of (iv) hold then equation (4) and the equalities \(M = M \Box \mathcal{J}(N)\) and \(N = \mathcal{I}(M) \Box N\) yield the equations \(M \Box N = (\mathcal{I}(M) \Box N) \ast M\) and \(M \Box N = (M \Box \mathcal{J}(N)) \ast N\), from which the conclusions of (iv) follow. \(\Box\)

Let \(\text{Lad}\) be the 2-dimensional category with objects small \(\mathcal{V}\)-categories \(C\) and \(\text{Lad}(C, C')\) the category of \(\mathcal{V}\)-functors from \(C\) to \(C'\) which are \(\mathcal{V}\)-left adjoints, i.e., \(\mathcal{V}\)-cocontinuous, with maps \(\mathcal{V}\)-natural transformations. There is a strict homomorphism of bicategories \(\Phi: B \to \text{Lad}\) defined by

\[
\Phi(C) = C \quad \text{and} \quad \Phi\left(C_1 \begin{array}{c} d_1 \\ \downarrow \end{array} C \begin{array}{c} d_0 \\ \downarrow \end{array} C_0\right) = \text{Ran}_{V^d_0} V^{d_1} = V^{\ast_{V^d_0}},
\]

where we have computed the right Kan extension in terms of \((V^{d_0})^t\), the left adjoint of \(V^{d_0}\). Note that

\[
\Phi(M)(C_0(X, -))(Y) = C(X, Y)
\]

for \(X\) in \(C_0\) and \(Y\) in \(C_1\). We then have natural transformations

\(
\Phi(\lambda_M): \Phi(M) \cdot \Phi(M^t) \to \Phi(\mathcal{I}(M)) = \text{id}_{\mathcal{V}_C 1}
\)

and

\(
\Phi(\rho_M): \Phi(M^t) \cdot \Phi(M) \to \Phi(\mathcal{J}(M)) = \text{id}_{\mathcal{V}_C 0}
\)
and for $X$ and $Z$ in $C_0$

$$\Phi(\rho_M)(C_0(X, -))(Z) = \rho_M(X, Z).$$

**Theorem 4.** If

$$M = \left( \begin{array}{cc} d_1 & d_0 \\ C_1 & C_0 \end{array} \right)$$

is a Morita context with a right $\ast$-inverse, then the following are true:

(i) $\Phi(M')$ is left adjoint to $\Phi(M)$ with counit $\Phi(\rho_M)$ and unit $\Phi(\lambda_M)^{-1}$ which is an isomorphism.

(ii) The functor $\Phi(M')$ maps $\mathbf{V}$-atoms [5, (4.3)] into $\mathbf{V}$-atoms and hence representables into $\mathbf{V}$-atoms.

(iii) The functor

$$\mathbf{C}^0 \xrightarrow{R} \mathbf{V} \xrightarrow{\Phi} \mathbf{V}^0,$$

where $R$ is the Yoneda embedding, is $\mathbf{V}$-full and faithful, i.e., $d_0 : C_0 \rightarrow C$ is $\mathbf{V}$-codense [4].

(iv) $M$ has a left $\ast$-inverse if and only if

$$\mathbf{C}^0 \xrightarrow{R_1} \mathbf{V} \xrightarrow{\Phi(M')} \mathbf{V}^0$$

is $\mathbf{V}$-dense.

**Proof.** (i) is a consequence of applying $\Phi$ which is a strict map of bicategories to equation (1) for $M$ and $M'$. (ii) follows from the fact that $\Phi(M)$ is the right adjoint of $\Phi(M')$ and is $\mathbf{V}$-cocontinuous. Thus for $G$ a $\mathbf{V}$-atom in $\mathbf{V}^C$, we have

$$\mathbf{V}^{C_0}(\Phi(M')G, -) \simeq \mathbf{V}^{C_1}(G, -) \cdot \Phi(M)$$

which is $\mathbf{V}$-cocontinuous. Part (iii) is equivalent to stating both that

$$\mathbf{C}^0 \xrightarrow{R_1} \mathbf{V} \xrightarrow{\Phi} \mathbf{V}^0$$

is $\mathbf{V}$-full and faithful, which is true since $R_1$ and $\Phi(M')$ are, and that

$$\Phi(M)(C_0(C_0, -))C_1 = \mathbf{V}^{C_0}(\Phi(M')R_1(C_1), C_0(C_0, -))$$

for $C_0$ in $C_0$ and $C_1$ in $C_1$, which holds since $\Phi(M)$ is $\mathbf{V}$-right adjoint to $\Phi(M')$.

To show (iv) we note that since $\Phi(M')$ is $\mathbf{V}$-cocontinuous, $\Phi(M') \cdot R_1$ is $\mathbf{V}$-dense if and only if $\Phi(M')$ is $\mathbf{V}$-dense. But $\Phi(M')$ is $\mathbf{V}$-dense if and
only if its right adjoint is full and faithful, i.e., $\Phi(\rho_M)$ is an isomorphism. But by (6) $\Phi(\rho_M)$ is an isomorphism if and only if $\rho_M$ is one. □

Corollary 5. The map of bicategories $\Phi: B \rightarrow \text{Lad}$ induces an isomorphism of the Picard groupoids [3, p. 57]

$$\hat{\Phi}: \text{Pic } B \rightarrow \text{Pic(} \text{Lad}).$$ □

$(\text{Pic } B)(C_0, C_1)$ is the set of isomorphism classes of Morita equivalences, i.e., invertible arrows from $C_0$ to $C_1$.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92664