CONVERGENT NETS OF PARABOLIC
AND GENERALIZED SUPERPARABOLIC FUNCTIONS

NEIL A. EKLUND

ABSTRACT. The well-known convergence properties of families of harmonic functions are generalized to functions which satisfy $Lu = 0$ where $L$ is the weak parabolic operator in divergence form. Properties of superharmonic functions are obtained for generalized superparabolic functions. These results are obtained on any bounded domain in $E^{n+1}$.

Consider the parabolic operator in divergence form given by

$$Lu = u_t - \sum_{i,j} a_{ij}(x, t)u_{x_i} + \sum_{j} d_j(x, t)u_{x_j} - b_i(x, t)u_{x_i} - c(x, t)u.$$

In two preceding papers by the author [4], [5], existence, representation, and a maximum principle were obtained for solutions of $Lu = 0$ in the cylindrical domain $Q = \Omega \times (0, T)$ for $\Omega \subset E^n$, and generalized superparabolic functions in $Q$ were introduced. In this article the author will consider convergent nets of parabolic and superparabolic functions on a bounded domain $U$ assumed to be in $E^n \times (0, T)$. Since some of the properties obtained in this article will depend on results for superparabolic functions, it is necessary to restate these results for the domain $U$. The numbering of definitions and theorems in this article continues from those in [5].

Definition 7. Let $z = (x, t), w = (y, s) \in U$. $z \prec w$ in $U$ if there is a polygonal path $\{C_{z,w}(a)\}$, $0 < a < 1$, such that

1. $C_{z,w}(0) = \{z\}$, $C_{z,w}(1) = \{w\}$,
2. if $C_{z,w}(\alpha) = \{\xi^\alpha, \tau^\alpha\}$, then $\alpha < \beta$ implies $\tau^\alpha < \tau^\beta$,
3. $|C_{z,w}(\alpha); 0 \leq \alpha \leq 1| \subset U$.

Note that Definitions 4, 5, and 6 can be generalized to the domain $U$ since all properties are in terms of standard rectangles in the given domain. Accordingly, let $L_U$, $L'_U$, $L''_U$, and $L'''_U$ denote the corresponding spaces. The...
Theorems and corollaries stated below correspond to the analogous results obtained for $Q$ in [5].

**Theorem 4'.** If $u \in \mathcal{S}_U^w$ and if there is a $z_0 \in U$ such that $0 \geq u(z_0) = \inf_U u$, then $u(z) = u(z_0)$ for all $z < z_0$ in $U$.

The following corollary is the analogue of that preceding Theorem 7 in [5].

**Corollary.** If $u \in \mathcal{S}_U^t$, then

1. $u(z_0) < \infty$ implies $u(z) < \infty$ for all $z < z_0$ in $U$;
2. $u(z_0) = \infty$ implies $u(z) = \infty$ for all $z < z_0$ in $U$.

**Theorem 10'.** Let $u \in \mathcal{S}_U^t$ and let $R$ be a standard rectangle in $U$. If $u < \infty$ on $R$, then $L(u; z, R)$ exists and the function

$$v(z) = \begin{cases} u(z), & z \in U - R, \\ L(u; z, R), & z \in R, \end{cases}$$

satisfies $u \geq v$, $Lv = 0$ on $R$, and $v \in \mathcal{S}_U^t$.

**Lemma 2.** If $\{u_\alpha; \alpha \in A\}$ is a net of functions parabolic on $U$ which converges uniformly on $U$ to $u$, then $u$ is parabolic on $U$.

**Proof.** For any standard rectangle $R$, $\overline{R} \subset U$, and any $z \in R$, $u_\alpha(z) = L(u_\alpha; z, R)$. Since $u_\alpha \to u$ uniformly on $R$, it follows that $u(z) = L(u; z, R)$. Since $R$ in $U$ was arbitrary, $u$ is parabolic on $U$.

**Theorem 11.** If $\mathcal{F}$ is a family of parabolic functions on $U$ uniformly bounded on a neighborhood of each point of $U$ and if $K$ is compact in $U$, then the family $\mathcal{F}$ is uniformly equicontinuous on $K$ and each net $\{u_\alpha; \alpha \in A\}$ in $\mathcal{F}$ has a subnet which converges uniformly on $K$. If $\{u_\alpha; \alpha \in A\}$ is a convergent net of uniformly bounded parabolic functions on $U$, then $\lim_A u_\alpha$ is parabolic on $U$.

**Proof.** For each $z \in K$, let $N_z$ denote the neighborhood of $z$ on which the family $\mathcal{F}$ is uniformly bounded. Then $\bigcup_{z \in K} N_z$ is an open cover for $K$ and, by the Heine-Borel theorem, there is a finite subcover $N$. Let $N = \bigcup_{j=1}^k \{N_{z_j}; z = z_j \in K\}$. Then the family $\mathcal{F}$ is uniformly bounded on $N$ (and hence on $K$). Each member of $\mathcal{F}$ is parabolic on $U$ and, hence, is Hölder continuous on $U$. Since $K$ is compact in $U$, the Hölder coefficients and exponents can be made uniform for the family $\mathcal{F}$ and it follows that $\mathcal{F}$ is an equicontinuous family on $K$. It follows from Arzela's theorem that each net $\{u_\alpha; \alpha \in A\}$ in $\mathcal{F}$ has a uniformly convergent subnet on $K$.

Let $\{u_\alpha; \alpha \in K\}$ be a convergent net of uniformly bounded parabolic
functions on $U$. Then $u \equiv \lim_A u_\alpha$ exists. Let $\{K_j\}$ be a sequence of compact sets in $U$ such that $K_j \uparrow U$. It follows from what has been shown that for $K_1$ there is a subnet $\{u_\alpha\}_1$ of $\{u_\alpha; \alpha \in \Lambda\}$ which converges to a parabolic function on $K_1$. Since $u_\alpha \rightarrow u$ on $U$, $u$ is parabolic on $K_1$. Using what has been shown on $K_2$, there is a subnet $\{u_\alpha\}_2$ of $\{u_\alpha\}_1$ which converges to a parabolic function, $u$, on $K_2$. Continuing this process, the diagonalization process can be used to see that $u$ is parabolic on all compact subsets of $U$ and, hence, on $U$ itself.

Definition 8. Let $u$ be defined on $U$. Let $U_P = \{z \in U; u$ locally parabolic at $z\}$ and $U_\pm = \{z \in U; u(z) = \pm \infty\}$. $u$ is said to have property $P_\pm$ if
1. $w \in U_P$ implies $z \in U_P$ for all $z \in U$ with $z \prec w$,
2. $w \in U_\pm$ implies $z \in U_\pm$ for all $z \in U$ with $z \succ w$.
3. $U - U_P - U_+ - U_-$ is a finite union of sets of the form $\Omega_f \times (t_j)$.

Theorem 12. If $u_j$ is an increasing (or decreasing) sequence of parabolic functions on $U$, then $u(z) = \lim u_j(z)$ satisfies property $P_+$ (or $P_-$).

Proof. Assume $u_j \uparrow u$. If for some $w \in U$, $u(w) < \infty$, then for any standard rectangle $R$ with $R \subset U$ and $w \in R$, the monotone convergence theorem implies that
1. \[ L(u; w, R) = \lim L(u_j; w, R) = \lim u_j(w) = u(w) < \infty \]
and, hence, $u(z) < \infty$ on $\partial_p R$ with $z \prec w$. Therefore, $U_P$ satisfies property (1) of Definition 8. Since $u \geq u_1 \geq -\infty$ by the parabolicity of $u_1$, $U_- = \emptyset$. To see that $U_+$ satisfies property (2), suppose $u(w) = +\infty$ and suppose for some $z \in U$ with $w \prec z$ that $u(z) < \infty$. Then by our first argument $u(w) < \infty$ and a contradiction is obtained. Therefore property (2) is satisfied. Since $u_j \uparrow u$, for any $z \in U$ either $u(z) < \infty$ or $u(z) = \infty$ must hold. Hence, property (3) is vacuously satisfied.

Definition 9. A family $\mathcal{F}$ of functions defined on $U$ is right-directed if for each pair $u, v \in \mathcal{F}$ there is a function $w \in \mathcal{F}$ such that $u \leq w$ and $v \leq w$. $\mathcal{F}$ is left-directed if the above inequalities are reversed.

Lemma 3. If $\{u_i; i \in I\}$ is a right-directed (left-directed) family of functions parabolic on $U$, then $u \equiv \sup_i u_i$ ($u \equiv \inf_i u_i$) satisfies property $P_+$ ($P_-$) in $U$.

Proof. If $\{u_i; i \in I\}$ contains only one function then we are done. If
\( \{u_i; i \in I\} \) contains two or more functions, then the right-directedness can be used to obtain a sequence \( \{u_j\} \) in \( \{u_i; i \in I\} \) such that \( u_j \uparrow u \equiv \sup_{\alpha < \beta} u_{\alpha} \). The desired result follows from Theorem 12.

**Definition 10.** A family \( \mathcal{F} \) in \( S_U' \) is saturated if

1. \( u, v \in \mathcal{F} \Rightarrow u \wedge v \in \mathcal{F} \),
2. \( u \in \mathcal{F} \Rightarrow u^* \in \mathcal{F} \) where, for some standard rectangle \( R \) in \( U \),

\[
 u^*(z) = \begin{cases} 
 u(z) & \text{in } U - R, \\
 L(u; z, R) & \text{in } R.
\end{cases}
\]

**Theorem 13.** If \( \mathcal{F} \) is a saturated family in \( S_U' \), then \( v = \inf_{\mathcal{F}} u \) satisfies property \( P_- \) in \( U \).

**Proof.** Let \( R \) be an arbitrary standard rectangle. It will be shown that \( v \) satisfies property \( P_- \) on \( R \) and the desired result will follow from the arbitrariness of \( R \) in \( U \).

For each \( u \in \mathcal{F} \), let \( u^* \) be defined as in (2). By Theorem 10', \( u^* \leq \inf_{\mathcal{F}} u \) on \( R \). Moreover, since \( \mathcal{F} \) is saturated in \( S_U' \), \( u \in \mathcal{F} \) implies \( u^* \in \mathcal{F} \). Therefore,

\[
v = \inf \{u; u \in \mathcal{F}\} = \inf \{u^*; u \in \mathcal{F}\}.
\]

Let \( \mathcal{F}^* = \{u^*; u \in \mathcal{F}\} \). If it can be shown that \( \mathcal{F}^* \) is a left-directed family, the desired result will follow from Lemma 3. Let \( u, w \in \mathcal{F} \). Then \( u \wedge w \in \mathcal{F} \) and hence \( u^*, w^* \), and \( (u \wedge w)^* \in \mathcal{F} \). However \( u \wedge w \leq u \) implies \( (u \wedge w)^* \leq u^* \) and similarly \( (u \wedge w)^* \leq w^* \). That is, for any two elements \( u^*, w^* \in \mathcal{F}^* \), there is an element, namely \( (u \wedge w)^* \), in \( \mathcal{F} \) such that \( (u \wedge w)^* \leq u^* \) and \( (u \wedge w)^* \leq w^* \). Hence, \( \mathcal{F}^* \) is left-directed and the proof is complete.

The preceding results for families of parabolic and generalized superparabolic functions will next be used to find the greatest parabolic minorant of a given function \( u \) on \( U \). Let \( W \) be an open set in \( U \) and let \( u \in \mathcal{S}_U' \). Let \( \{R_j\} \) be a sequence of standard rectangles satisfying

1. \( \overline{R}_j \subset W \) for all \( j \),
2. \( W = \bigcup_{j=1}^{\infty} R_j \),
3. each rectangle \( R_j \) occurs infinitely often in the sequence \( \{R_j\} \).

Define

\[
u_1(z) = \begin{cases} 
 u(z), & z \in U - R_1, \\
 L(u; z, R_1), & z \in R_1,
\end{cases}
\]

and inductively
Then by Theorem 10', \( u_n < u_{n-1} \) on \( U \), \( u_n \) is parabolic on \( R_n \), and \( u_n \in S_U^\gamma \). Since \( \{u_n\} \) is a decreasing sequence, \( u_\infty(z) \equiv \lim_{n \to \infty} u_n(z) \) satisfies property \( P_- \) on \( U \).

**Definition 11.** \( u_\infty \) is called the reduction of \( u \) over \( W \) relative to \( U \).

It appears as if for a given open set \( W \) in \( U \) the corresponding reduction \( u_\infty \) will depend on the defining sequence \( \{R_j\} \). It will follow from Theorem 15 that \( u_\infty \) is independent of the choice of \( \{R_j\} \) if there is a parabolic function \( v \) on \( W \) with \( v \leq u \).

**Theorem 14.** If \( u \in S_U^\gamma \), \( W \) is open in \( U \), and \( u_\infty \) is the reduction of \( u \) over \( W \) relative to \( U \) corresponding to the sequence \( \{R_j\} \), then \( u_\infty \) satisfies property \( P_- \) on each component of \( W \), and \( u_\infty = u \) on \( U - W \).

**Proof.** Without loss of generality assume \( W \) is connected. For each \( j \), there is a sequence \( \{j^k\} \) such that \( R_{j^k} = R_j \) for all \( k \). Thus, on \( R_j \),

\[
u_\infty(z) = \lim_{k \to \infty} \nu_{j^k}(z).
\]

Since \( \nu_{j^k} \) is parabolic on \( R_j \) for all \( k \), it follows from Theorem 12 that \( u_\infty \) satisfies property \( P_- \) on \( R_j \). Using this procedure on each different rectangle \( R_j \) in the defining sequence, it follows that \( u_\infty \) satisfies property \( P_- \) on \( W \). Since \( u_j \equiv u \) on \( U - W \) for all \( j \), \( u_\infty = u \) on \( U - W \).

**Lemma 4.** If \( u \in S_U^\gamma \), \( W \) is open in \( U \), \( v \) is parabolic on \( W \), and \( v \leq u \) on \( W \), then \( v \leq u_\infty \leq u \) on \( W \) where \( u_\infty \) is the reduction of \( u \) over \( W \) in \( U \).

**Proof.** Let \( \{R_j\} \) be the sequence of standard rectangles which define the reduction of \( u \) over \( W \), \( u_\infty \). Then

\[
u(z) = L(v; z, R_1) \leq L(u; z, R_1) = u_1(z) \leq u(z) \quad \text{on } R_1,
\]

and

\[
u(z) \leq u(z) = u_1(z) \quad \text{on } W - R_1.
\]

Therefore, \( v \leq u_1 \leq u \) on \( W \). Proceeding inductively, the desired result is obtained.

**Definition 12.** If \( u \in S_U^\gamma \), \( v \) is parabolic on \( U \), and \( v \leq u \) on \( U \), then \( v \) is called a parabolic minorant of \( u \). \( v \) is the greatest parabolic minorant of \( u \) if \( v \) is a parabolic minorant of \( u \) and any other parabolic minorant of
Theorem 15. If \( v \in S'_U \), \( W \) is open in \( U \), and \( u \) has a parabolic minorant \( v \) on \( W \), then \( u \) has a unique greatest parabolic minorant on \( W \); namely, \( u_\infty \).

Proof. It follows from Lemma 4 that \( v \leq u_\infty \leq u \) on \( W \). But \( v \) parabolic on \( W \) implies \( v > -\infty \) on \( W \). Therefore, since \( u_\infty \) satisfies property \( P \) on \( W \), \( u_\infty \) is parabolic on \( W \). If there were two such \( u_\infty \), say \( u_{\infty,1} \) and \( u_{\infty,2} \), then \( u_{\infty,1} \leq u_{\infty,2} \leq u \) on \( W \) since \( u_{\infty,1} \) is a parabolic minorant and \( u_{\infty,2} \leq u_{\infty,1} \leq u \) on \( W \) since \( u_{\infty,2} \) is a parabolic minorant. Therefore, \( u_\infty \) is unique and it is the greatest parabolic minorant of \( u \).

Theorem 16. If \( u, v \in S'_U \) have harmonic minorants, then \( (u + v)_\infty = u_\infty + v_\infty \).

Proof. Since \( u \) and \( v \) have harmonic minorants, \( u + v \) does also and, hence, \( u_\infty, v_\infty \), and \( (u + v)_\infty \) are independent of the defining sequence \( \{R_j\} \). For such a sequence and \( j \geq 1 \)

\[
(u + v)_j (x) = \begin{cases} 
(u + v)_{j-1} (z) & \text{on } U - R_j, \\
L((u + v)_{j-1}; z, R_j) & \text{on } R_j,
\end{cases}
\]

\[
= \begin{cases} 
u_{j-1} (z) & \text{on } U - R_j, \\
L(v_{j-1}; z, R_j) & \text{on } R_j,
\end{cases} + \begin{cases} u_{j-1} (z) & \text{on } U - R_j, \\
L(u_{j-1}; z, R_j) & \text{on } R_j.
\end{cases}
\]

\[= u_j (z) + v_j (z).\]

Since \( u_j \downarrow u_\infty \) and \( v_j \downarrow v_\infty \), it follows that \( (u + v)_j \downarrow (u + v)_\infty \) and \( (u + v)_\infty = u_\infty + v_\infty \).

REFERENCES


DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37235

Current address: Department of Mathematics, Centre College, Danville, Kentucky 40422