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GROUPS WITH FAITHFUL BLOCKS

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ABSTRACT. A necessary and sufficient condition is given for a finite group to have a p-block with kernel $\{1\}$. This extends a theorem of Gaschütz on the existence of a faithful irreducible representation of a finite group.

Let G be a finite group and F a field of characteristic $p \geq 0$. Then the group algebra FG is a direct sum of two-sided ideals B_i which are indecomposable as two-sided ideals $FG = B_1 \oplus \cdots \oplus B_n$. The B_i are called the block ideals of FG. If e is a block idempotent, i.e. the identity element of a block ideal B, then a (left) FG-module V is said to belong to the block $B \leftrightarrow e$ if $eV = V$.

Furthermore, if $p > 0$ and R is a complete discrete valuation ring with residue class field F and quotient field K of characteristic 0, then every block idempotent e can be lifted to a block idempotent $\check{e}$ of RG, which can be embedded in KG. A KG-module V is also said to belong to the block $B \leftrightarrow e$ if $\check{e}V = V$.

For the case that F and K are splitting fields for G, R. Brauer [1] has defined the kernel of the block $B \leftrightarrow e$ to be the intersection of the kernels of the (ordinary) irreducible K-representations of G belonging to $B \leftrightarrow e$. Returning to the case where F is an arbitrary field we feel that it is natural to make the following

Definition. The kernel $N(e)$ of a block $B \leftrightarrow e$ is the kernel of the F-representation of G, which is afforded by the block ideal B. Thus $N(e) = \{g \in G | ge = e\}$.

We remark that this agrees with Brauer's definition as Proposition 1(b) below shows, but differs slightly from the one used in [3] and [4], where the intersection $N^*(e)$ of the kernels of the irreducible F-representations of G belonging to $B \leftrightarrow e$ was called the kernel of $B \leftrightarrow e$. The kernel $N(e)$ is uniquely determined by $N^*(e)$ and vice versa. In fact, by a result of

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Michler [3]

$$N(e) = O_{p'}(N^*(e)), \qquad N^*(e)/N(e) = O_p(G/N(e)),$$

i.e. $N(e)$ is the maximal p-regular normal subgroup of $N^*(e)$ and $N^*(e)/N(e)$ is the maximal normal p-subgroup of $G/N(e)$. Here a normal subgroup is called p-regular, if its order is not divisible by p.

Proposition 1. (a) *The kernel $N(e)$ of the block $B \leftrightarrow e$ is equal to the kernel of every principal indecomposable module belonging to $B \leftrightarrow e$.*

(b) *If $p > 0$ and R is a complete discrete valuation ring with residue class field F and quotient field K of characteristic 0, then $N(e)$ is the intersection of the kernels of the irreducible K-representations belonging to $B \leftrightarrow e$.*

Proof. (a) Obviously $N(e)$ is the intersection of the kernels of the principal indecomposable F-representations belonging to $B \leftrightarrow e$. Let H be the kernel of a principal indecomposable module FGu belonging to $B \leftrightarrow e$, where u denotes a primitive idempotent of FG. H must be p-regular, since if $u = \Sigma_{g \in G} \alpha_g g$, then α_g is constant on the cosets of H, because for $h \in H$, $hu = u$. Thus $u = \Sigma_{i=1}^r \alpha_i g_i(\Sigma_{h \in H} h)$, where $g_1, \ldots, g_r$ are coset representatives of H in G, and

$$u = u^2 = |H| \sum_{i,j} \alpha_i \alpha_j g_i g_j \left(\sum_{h \in H} h \right),$$

which would be 0 if p divides $|H|$.

Hence one can form $s_H = |H|^{-1} \Sigma_{h \in H} h$ and one has $s_H u = u$. But s_H is a central idempotent and $s_H = \Sigma_{N(e_i) \geq H} e_i$. Since $s_H u = u$, the block idempotent e must occur in the sum, therefore $N(e) \geq H$.

(b) $KG\check{e}$ is the direct sum of those simple ideals, which have irreducible modules belonging to $B \leftrightarrow e$. Hence the intersection of the kernels of the irreducible K-representations belonging to $B \leftrightarrow e$ is $\{g \in G | g\check{e} = \check{e}\}$. Since $\check{e}$ is mapped onto e under the natural map $RG \to FG$, one has obviously $\{g \in G | g\check{e} = \check{e}\} \leq N(e)$. Conversely, let $x \in N(e)$, hence $xe = e$. We denote the unique maximal ideal of R by πR. If $x\check{e} \neq \check{e}$, there is a maximal number $k \geq 1$ such that $x\check{e} - \check{e} \in \pi^k RG$. Let $\check{e} = \Sigma_{g \in G} \alpha_g g$ ($\alpha_g \in R$); then $\alpha_g \equiv \alpha_{xg} \equiv \cdots \equiv \alpha_{x^{m-1}g} \mod(\pi^k)$, where m is the order of x. Hence, if $g_1, \ldots, g_r$ are coset representatives of $\langle x \rangle$ in G,

$$e = \sum_{i=1}^{r} (\alpha_i(1 + x + \cdots + x^{m-1}) + b_i)g_i$$

with $b_i \in \pi^k RG$ and $\alpha_i = \alpha_{g_i}$. But then

$$x\check{e} - \check{e} = (x\check{e} - \check{e})\check{e} = \sum_{i=1}^{r} (x\check{e} - \check{e}) b_i g_i \in \pi^{2k} RG,$$

a contradiction.

The natural question—Which p-regular normal subgroups H can be kernels of blocks?—can be reduced to the case $H = \{1\}$. FG has a block with kernel H if and only if H is p-regular and $F(G/H)$ has a block with kernel $\{1\}$. This follows from the fact that if H is p-regular and $s_H = |H|^{-1}\Sigma_{h \in H} h$, then $FG = FGs_H \oplus FG(1 - s_H)$, where $FGs_H \cong F(G/H)$ is the direct sum of all block ideals of FG with kernels containing H.

Proposition 2. *FG has a block with kernel $\{1\}$ if and only if the maximal p-regular normal subgroup $O_{p'}(\mathrm{Soc}\,(G))$ of the socle of G is generated by one class of conjugate elements of G.*

For the case $p = 0$ or p a prime not dividing the order of G this proposition contains the theorem of Gaschütz [2] (see also Žmud' [5]) on the existence of a faithful irreducible representation; for in this case the kernel of a block is simply the kernel of an irreducible representation, and $O_{p'}(\mathrm{Soc}\,(G)) = \mathrm{Soc}\,(G)$.

Proof of Proposition 2. It was shown in [4] that if FG has a block with kernel $\{1\}$ then the same is true also for $F'G$, where F' is any field with the same characteristic as F. Hence one can assume that F is a splitting field for G.

Let Φ_i $(1 \le i \le r)$ be the characters of the principal indecomposable representations of FG and $\phi_i(1)$ be the degrees of the corresponding irreducible representations of FG. If H is a p-regular normal subgroup of G, it follows from the orthogonality relations for G/H that

$$\sum_{N(\Phi_i) \ge H} \phi_i(1)\Phi_i(x) = \begin{cases} |G/H| & \text{if } x \in H, \\ 0 & \text{if } x \notin H, \end{cases}$$

where the sum ranges over all i such that the kernel $N(\Phi_i)$ contains H.

If $M_1, \ldots, M_m$ are the minimal normal p-regular subgroups of G, then

$$\psi(x) = \sum_{N(\Phi_i) = \{1\}} \phi_i(1)\Phi_i(x)$$

$$= \sum_{i=1}^{r} \phi_i(1)\Phi_i(x) + \sum_{k=1}^{m} (-1)^k \sum_{1 \le j_1 < \cdots < j_k \le m} \; \sum_{N(\Phi_i) \ge M_{j_1} \cdots M_{j_k}} \phi_i(1)\Phi_i(x),$$

where an empty sum is understood to be 0. Hence

$$\psi(1) = |G| + \sum_{k=1}^{m} (-1)^k \sum_{1 \le j_1 < \cdots < j_k \le m} |G: M_{j_1} \cdots M_{j_k}|.$$

The lattice of normal subgroups of G, which are contained in $Q = O_{p'}(\mathrm{Soc}(G))$ has a duality ϵ such that $(MN)^\epsilon = M^\epsilon \cap N^\epsilon$, $(M \cap N)^\epsilon = M^\epsilon N^\epsilon$ and $|N^\epsilon| = |Q:N|$. If $N_1, \ldots, N_m$ are the normal subgroups of G which are maximal in Q, then

$$\psi(1) = |G| + \sum_{k=1}^{m} (-1)^k \sum_{1 \le j_1 < \cdots < j_k \le m} |G: N_{j_1}^\epsilon \cdots N_{j_k}^\epsilon|.$$

Since $|Q:N_{j_1}^\epsilon \cdots N_{j_k}^\epsilon| = |N_{j_1} \cap \cdots \cap N_{j_k}|$, it follows that

$$\psi(1) = |G:Q| \left| Q \setminus \bigcup_{j=1}^{m} N_j \right|,$$

and this is different from zero if and only if Q is generated by one class of conjugate elements of G. Q.E.D.

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