GROUPS WITH FAITHFUL BLOCKS

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ABSTRACT. A necessary and sufficient condition is given for a finite group to have a p-block with kernel \{1\}. This extends a theorem of Gaschütz on the existence of a faithful irreducible representation of a finite group.

Let \( G \) be a finite group and \( F \) a field of characteristic \( p \geq 0 \). Then the group algebra \( FG \) is a direct sum of two-sided ideals \( B_i \) which are indecomposable as two-sided ideals \( FG = B_1 \oplus \cdots \oplus B_n \). The \( B_i \) are called the block ideals of \( FG \). If \( e \) is a block idempotent, i.e. the identity element of a block ideal \( B_i \), then a (left) \( FG \)-module \( V \) is said to belong to the block \( B \leftrightarrow e \) if \( eV = V \).

Furthermore, if \( p > 0 \) and \( R \) is a complete discrete valuation ring with residue class field \( F \) and quotient field \( K \) of characteristic 0, then every block idempotent \( e \) can be lifted to a block idempotent \( \bar{e} \) of \( RG \), which can be embedded in \( KG \). A \( KG \)-module \( V \) is also said to belong to the block \( B \leftrightarrow e \) if \( \bar{e}V = V \).

For the case that \( F \) and \( K \) are splitting fields for \( G \), R. Brauer \[1\] has defined the kernel of the block \( B \leftrightarrow e \) to be the intersection of the kernels of the (ordinary) irreducible \( K \)-representations of \( G \) belonging to \( B \leftrightarrow e \). Returning to the case where \( F \) is an arbitrary field we feel that it is natural to make the following

Definition. The kernel \( N(e) \) of a block \( B \leftrightarrow e \) is the kernel of the \( F \)-representation of \( G \), which is afforded by the block ideal \( B \). Thus \( N(e) = \{ g \in G | ge = e \} \).

We remark that this agrees with Brauer's definition as Proposition 1(b) below shows, but differs slightly from the one used in \[3\] and \[4\], where the intersection \( N^*(e) \) of the kernels of the irreducible \( F \)-representations of \( G \) belonging to \( B \leftrightarrow e \) was called the kernel of \( B \leftrightarrow e \). The kernel \( N(e) \) is uniquely determined by \( N^*(e) \) and vice versa. In fact, by a result of

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Michler [3]

\[ N(e) = O_p(N^*(e)), \quad N^*(e)/N(e) = O_p(G/N(e)), \]

i.e. \( N(e) \) is the maximal \( p \)-regular normal subgroup of \( N^*(e) \) and \( N^*(e)/N(e) \) is the maximal normal \( p \)-subgroup of \( G/N(e) \). Here a normal subgroup is called \( p \)-regular, if its order is not divisible by \( p \).

Proposition 1. (a) The kernel \( N(e) \) of the block \( B \leftrightarrow e \) is equal to the kernel of every principal indecomposable module belonging to \( B \leftrightarrow e \).

(b) If \( p > 0 \) and \( R \) is a complete discrete valuation ring with residue class field \( F \) and quotient field \( K \) of characteristic 0, then \( N(e) \) is the intersection of the kernels of the irreducible \( K \)-representations belonging to \( B \leftrightarrow e \).

Proof. (a) Obviously \( N(e) \) is the intersection of the kernels of the principal indecomposable \( F \)-representations belonging to \( B \leftrightarrow e \). Let \( H \) be the kernel of a principal indecomposable module \( FGu \) belonging to \( B \leftrightarrow e \), where \( u \) denotes a primitive idempotent of \( FG \). \( H \) must be \( p \)-regular, since if \( u = \sum_{g \in G} \alpha_g g \), then \( \alpha_g \) is constant on the cosets of \( H \), because for \( b \in H \), \( bu = u \). Thus \( u = \sum_{i=1}^r \alpha_{i,b} g_i \), where \( g_1, \ldots, g_r \) are coset representatives of \( H \) in \( G \), and

\[ u = u^2 = |H| \sum_{i,j} \alpha_{i,b} \alpha_{j,b} g_i \left( \sum_{b \in H} h \right), \]

which would be 0 if \( p \) divides \( |H| \).

Hence one can form \( s_H = |H|^{-1} \sum_{b \in H} h \) and one has \( s_H u = u \). But \( s_H \) is a central idempotent and \( s_H = \sum_{N(e) \supseteq H} e_i \). Since \( s_H u = u \), the block idempotent \( e \) must occur in the sum, therefore \( N(e) \geq H \).

(b) \( KG\hat{e} \) is the direct sum of those simple ideals, which have irreducible modules belonging to \( B \leftrightarrow e \). Hence the intersection of the kernels of the irreducible \( K \)-representations belonging to \( B \leftrightarrow e \) is \( \{ g \in G | g\hat{e} = \hat{e} \} \). Since \( \hat{e} \) is mapped onto \( e \) under the natural map \( RG \rightarrow FG \), one has obviously \( \{ g \in G | g\hat{e} = \hat{e} \} \subseteq N(e) \). Conversely, let \( x \in N(e) \), hence \( xe = e \). We denote the unique maximal ideal of \( R \) by \( \pi R \). If \( xe \neq e \), there is a maximal number \( k > 1 \) such that \( xe - e \in \pi^k R G \). Let \( \hat{e} = \sum_{g \in G} \alpha_g g \) (\( \alpha_g \in R \)); then \( \alpha_{\hat{e}} = \alpha_{x\hat{e}} = \cdots = \alpha_{x^{m-1}g} \mod(\pi^k) \), where \( m \) is the order of \( x \). Hence, if \( g_1, \ldots, g_r \) are coset representatives of \( \langle x \rangle \) in \( G \),

\[ e = \sum_{i=1}^r (\alpha_{i,b} (1 + x + \cdots + x^{m-1}) + b_i) g_i \]

with \( b_i \in \pi^k R G \) and \( \alpha_{i,b} = \alpha_{i,x\hat{e}} \). But then
\[ x^e - e = (x^e - e)\bar{e} = \sum_{i=1}^{r} (x^e - e)b_i g_i \in \pi^{2k} RG, \]

a contradiction.

The natural question—Which \( p \)-regular normal subgroups \( H \) can be kernels of blocks?—can be reduced to the case \( H = \{1\} \). \( FG \) has a block with kernel \( H \) if and only if \( H \) is \( p \)-regular and \( F(G/H) \) has a block with kernel \( \{1\} \). This follows from the fact that if \( H \) is \( p \)-regular and \( s_H = |H|^{-1}\sum h \in H h \), then \( FG = FGs_H \oplus FG(1 - s_H) \), where \( FGs_H \cong F(G/H) \) is the direct sum of all block ideals of \( FG \) with kernels containing \( H \).

**Proposition 2.** \( FG \) has a block with kernel \( \{1\} \) if and only if the maximal \( p \)-regular normal subgroup \( O_p(Soc(G)) \) of the socle of \( G \) is generated by one class of conjugate elements of \( G \).

For the case \( p = 0 \) or \( p \) a prime not dividing the order of \( G \) this proposition contains the theorem of Gaschütz [2] (see also Žmud [5]) on the existence of a faithful irreducible representation; for in this case the kernel of a block is simply the kernel of an irreducible representation, and \( O_p(Soc(G)) = Soc(G) \).

**Proof of Proposition 2.** It was shown in [4] that if \( FG \) has a block with kernel \( \{1\} \) then the same is true also for \( E G \), where \( E \) is any field with the same characteristic as \( F \). Hence one can assume that \( F \) is a splitting field for \( G \).

Let \( \Phi_i \ (1 \leq i \leq r) \) be the characters of the principal indecomposable representations of \( FG \) and \( \phi_i(1) \) be the degrees of the corresponding irreducible representations of \( FG \). If \( H \) is a \( p \)-regular normal subgroup of \( G \), it follows from the orthogonality relations for \( G/H \) that

\[
\sum_{N(\Phi_i) \geq H} \phi_i(1)\Phi_i(x) = \begin{cases} |G/H| & \text{if } x \in H, \\ 0 & \text{if } x \not\in H, \end{cases}
\]

where the sum ranges over all \( i \) such that the kernel \( N(\Phi_i) \) contains \( H \).

If \( M_1, \ldots, M_m \) are the minimal normal \( p \)-regular subgroups of \( G \), then

\[
\psi(x) = \sum_{N(\Phi_i) = \{1\}} \phi_i(1)\Phi_i(x)
= \sum_{i=1}^{r} \phi_i(1)\Phi_i(x) + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq m} \sum_{N(\Phi_i) \geq M_{j_1} \cdots M_{j_k}} \phi_i(1)\Phi_i(x),
\]
where an empty sum is understood to be 0. Hence

$$\psi(1) = |G| + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq m} |G : M_{j_1} \cdots M_{j_k}|.$$ 

The lattice of normal subgroups of $G$, which are contained in $Q = O_p^*(\text{Soc}(G))$ has a duality $\epsilon$ such that $(MN)^\epsilon = M^\epsilon \cap N^\epsilon$, $(M \cap N)^\epsilon = M^\epsilon N^\epsilon$, and $|N^\epsilon| = |Q : N|$. If $N_1, \ldots, N_m$ are the normal subgroups of $G$ which are maximal in $Q$, then

$$\psi(1) = |G| + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq m} |G : N_{j_1}^\epsilon \cdots N_{j_k}^\epsilon|.$$ 

Since $|Q : N_{j_1}^\epsilon \cdots N_{j_k}^\epsilon| = |N_{j_1} \cap \cdots \cap N_{j_k}|$, it follows that

$$\psi(1) = |G : Q| \left| Q \bigcap \bigcup_{j=1}^{m} N_j \right|,$$

and this is different from zero if and only if $Q$ is generated by one class of conjugate elements of $G$. Q.E.D.

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