ABSTRACT. A necessary and sufficient condition is given for a finite group to have a $p$-block with kernel $\{1\}$. This extends a theorem of Gaschütz on the existence of a faithful irreducible representation of a finite group.

Let $G$ be a finite group and $F$ a field of characteristic $p > 0$. Then the group algebra $FG$ is a direct sum of two-sided ideals $B_i$ which are indecomposable as two-sided ideals $FG = B_1 \oplus \cdots \oplus B_n$. The $B_i$ are called the block ideals of $FG$. If $e$ is a block idempotent, i.e. the identity element of a block ideal $B$, then a (left) $FG$-module $V$ is said to belong to the block $B \leftrightarrow e$ if $eV = V$.

Furthermore, if $p > 0$ and $R$ is a complete discrete valuation ring with residue class field $F$ and quotient field $K$ of characteristic 0, then every block idempotent $e$ can be lifted to a block idempotent $\bar{e}$ of $RG$, which can be embedded in $KG$. A $KG$-module $V$ is also said to belong to the block $B \leftrightarrow e$ if $\bar{e}V = V$.

For the case that $F$ and $K$ are splitting fields for $G$, R. Brauer [1] has defined the kernel of the block $B \leftrightarrow e$ to be the intersection of the kernels of the (ordinary) irreducible $K$-representations of $G$ belonging to $B \leftrightarrow e$. Returning to the case where $F$ is an arbitrary field we feel that it is natural to make the following

Definition. The kernel $N(e)$ of a block $B \leftrightarrow e$ is the kernel of the $F$-representation of $G$, which is afforded by the block ideal $B$. Thus $N(e) = \{g \in G | ge = e\}$.

We remark that this agrees with Brauer's definition as Proposition 1(b) below shows, but differs slightly from the one used in [3] and [4], where the intersection $N^*(e)$ of the kernels of the irreducible $F$-representations of $G$ belonging to $B \leftrightarrow e$ was called the kernel of $B \leftrightarrow e$. The kernel $N(e)$ is uniquely determined by $N^*(e)$ and vice versa. In fact, by a result of

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Michler [3]

\[ N(e) = O_p(N^*(e)), \quad N^*(e)/N(e) = O_p(G/N(e)), \]

i.e. \( N(e) \) is the maximal \( p \)-regular normal subgroup of \( N^*(e) \) and \( N^*(e)/N(e) \) is the maximal normal \( p \)-subgroup of \( G/N(e) \). Here a normal subgroup is called \( p \)-regular, if its order is not divisible by \( p \).

**Proposition 1.** (a) The kernel \( N(e) \) of the block \( B \leftrightarrow e \) is equal to the kernel of every principal indecomposable module belonging to \( B \leftrightarrow e \).

(b) If \( p > 0 \) and \( R \) is a complete discrete valuation ring with residue class field \( F \) and quotient field \( K \) of characteristic 0, then \( N(e) \) is the intersection of the kernels of the irreducible \( K \)-representations belonging to \( B \leftrightarrow e \).

**Proof.** (a) Obviously \( N(e) \) is the intersection of the kernels of the principal indecomposable \( F \)-representations belonging to \( B \leftrightarrow e \). Let \( H \) be the kernel of a principal indecomposable module \( FGu \) belonging to \( B \leftrightarrow e \), where \( u \) denotes a primitive idempotent of \( FG \). \( H \) must be \( p \)-regular, since if \( u = \sum g \in G \alpha_g g \), then \( \alpha_g \) is constant on the cosets of \( H \), because for \( h \in H \), \( hu = u \). Thus \( u = \sum_{i=1}^r \alpha_i g_i (\sum h \in H h) \), where \( g_1, \ldots, g_r \) are coset representatives of \( H \) in \( G \), and

\[
0 = u^2 = |H| \sum_{i,j} \alpha_i \alpha_j g_i g_j \left( \sum h \in H h \right),
\]

which would be 0 if \( p \) divides \(|H|\).

Hence one can form \( s_H = |H|^{-1} \sum h \in H h \) and one has \( s_H u = u \). But \( s_H \) is a central idempotent and \( s_H = \sum_{N(e) \supseteq H} e_i \). Since \( s_H u = u \), the block idempotent \( e \) must occur in the sum, therefore \( N(e) \supseteq H \).

(b) \( KG\tilde{e} \) is the direct sum of those simple ideals, which have irreducible modules belonging to \( B \leftrightarrow e \). Hence the intersection of the kernels of the irreducible \( K \)-representations belonging to \( B \leftrightarrow e \) is \( \{ g \in G | g\tilde{e} = \tilde{e} \} \). Since \( \tilde{e} \) is mapped onto \( e \) under the natural map \( RG \to FG \), one has obviously \( \{ g \in G | g\tilde{e} = \tilde{e} \} \leq N(e) \). Conversely, let \( x \in N(e) \), hence \( xe = e \). We denote the unique maximal ideal of \( R \) by \( \pi R \). If \( x\tilde{e} \notin \tilde{e} \), there is a maximal number \( k \geq 1 \) such that \( x\tilde{e} - \tilde{e} \in \pi^k RG \). Let \( \tilde{e} = \sum g \in G \alpha_g g \) (\( \alpha_g \in R \)); then \( \alpha_g = \alpha x_g \equiv \cdots \equiv \alpha x_{m-1} g \mod (\pi^k) \), where \( m \) is the order of \( x \). Hence, if \( g_1, \ldots, g_r \) are coset representatives of \( \langle x \rangle \) in \( G \),

\[
e = \sum_{i=1}^r (\alpha_i (1 + x + \cdots + x^{m-1} + b_i) g_i)
\]

with \( b_i \in \pi^k RG \) and \( \alpha_i = \alpha g_i \). But then
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\[ x^e - e = (x^e - e)e = \sum_{i=1}^{r} (x^e - e)b_{i} \in \pi^{2k} RG, \]

a contradiction.

The natural question—Which \( p \)-regular normal subgroups \( H \) can be kernels of blocks?—can be reduced to the case \( H = \{1\} \). \( FG \) has a block with kernel \( H \) if and only if \( H \) is \( p \)-regular and \( F(G/H) \) has a block with kernel \( \{1\} \). This follows from the fact that if \( H \) is \( p \)-regular and \( s_{H} = |H|^{-1} \sum_{b \in H} b \), then \( FG = FG_{sH} \oplus FG(1 - s_{H}) \), where \( FG_{sH} \cong F(G/H) \) is the direct sum of all block ideals of \( FG \) with kernels containing \( H \).

Proposition 2. \( FG \) has a block with kernel \( \{1\} \) if and only if the maximal \( p \)-regular normal subgroup \( O_{p'}(\text{Soc}(G)) \) of the socle of \( G \) is generated by one class of conjugate elements of \( G \).

For the case \( p = 0 \) or \( p \) a prime not dividing the order of \( G \) this proposition contains the theorem of Gaschütz [2] (see also Žmud' [5]) on the existence of a faithful irreducible representation; for in this case the kernel of a block is simply the kernel of an irreducible representation, and \( O_{p'}(\text{Soc}(G)) = \text{Soc}(G) \).

Proof of Proposition 2. It was shown in [4] that if \( FG \) has a block with kernel \( \{1\} \) then the same is true also for \( F'G \), where \( F' \) is any field with the same characteristic as \( F \). Hence one can assume that \( F \) is a splitting field for \( G \).

Let \( \Phi_{i} \ (1 \leq i \leq r) \) be the characters of the principal indecomposable representations of \( FG \) and \( \phi_{i}(1) \) be the degrees of the corresponding irreducible representations of \( FG \). If \( H \) is a \( p \)-regular normal subgroup of \( G \), it follows from the orthogonality relations for \( G/H \) that

\[
\sum_{N(\Phi_{i}) \geq H} \phi_{i}(1)\Phi_{i}(x) = \begin{cases} |G/H| & \text{if } x \in H, \\ 0 & \text{if } x \not\in H, \end{cases}
\]

where the sum ranges over all \( i \) such that the kernel \( N(\Phi_{i}) \) contains \( H \).

If \( M_{1}, \ldots, M_{m} \) are the minimal normal \( p \)-regular subgroups of \( G \), then

\[
\psi(x) = \sum_{N(\Phi_{i}) = \{1\}} \phi_{i}(1)\Phi_{i}(x) = \sum_{i=1}^{r} \phi_{i}(1)\Phi_{i}(x) + \sum_{k=1}^{m} (-1)^{k} \sum_{1 \leq j_{1} \cdots < j_{k} \leq m} \sum_{N(\Phi_{j_{1}}) \geq M_{j_{1}} \cdots M_{j_{k}}} \phi_{j_{1}}(1)\Phi_{j_{1}}(x),
\]

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where an empty sum is understood to be 0. Hence
\[ \psi(1) = |G| + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq m} |G: M_{j_1} \cdots M_{j_k}|. \]

The lattice of normal subgroups of \( G \), which are contained in \( Q = O_p'(\text{Soc}(G)) \) has a duality \( \epsilon \) such that \((MN)\epsilon = M\epsilon \cap N\epsilon\), \((M \cap N)\epsilon = M\epsilon N\epsilon\), and \(|N\epsilon| = |Q:N|\). If \( N_1, \ldots, N_m \) are the normal subgroups of \( G \) which are maximal in \( Q \), then
\[ \psi(1) = |G| + \sum_{k=1}^{m} (-1)^k \sum_{1 \leq j_1 < \cdots < j_k \leq m} |G: N_{j_1}^\epsilon \cdots N_{j_k}^\epsilon|. \]

Since \(|Q:N_{j_1}^\epsilon \cdots N_{j_k}^\epsilon| = |N_{j_1} \cap \cdots \cap N_{j_k}|\), it follows that
\[ \psi(1) = |G:Q| \bigg| \bigcup_{j=1}^{m} N_j \bigg|, \]
and this is different from zero if and only if \( Q \) is generated by one class of conjugate elements of \( G \). Q.E.D.

REFERENCES

5. È. M. Žmud', On the kernels of homomorphisms of linear representations of a finite group, Mat. Sb. 44(86) (1958), 353–408. (Russian) MR 20 #5236.