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CONDITIONS ON A COMPACT CONNECTED LIE GROUP WHICH INSURE A "WEYL CHARACTER FORMULA"

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ABSTRACT. A theorem showing the equivalence of three conditions on a compact connected Lie group is proved. Among the corollaries is an extended "Weyl character formula" as originally stated by Bott.

In [3] Bott states a generalized "Weyl character formula" for compact connected Lie groups G whose fundamental group $\pi_1(G)$ has no 2-torsion (recall that the original formula is stated for G simply connected). As in the original character formula the critical issue surrounds the lifting of the adjoint representation Ad: $G \longrightarrow SO(n)$. For $\pi_1(G) = 0$ Ad can be lifted to $\mathrm{Spin}(n)$ where π: $\mathrm{Spin}(n) \longrightarrow SO(n)$ is the usual double cover. In the case where $\pi_1(G)$ has no 2-torsion, Ad can be lifted to $\mathrm{Spin}^c(n)$.

The purpose of this note is to prove that three properties of compact connected Lie groups are equivalent. A corollary is a proof of the assertion in [3, p. 178] that a character formula exists for any compact connected G where $\pi_1(G)$ has no 2-torsion. The Theorem also lays the groundwork for the generalization in [6] of a theorem originally proved in [4] and [5] for a large number of special cases; a "Poincaré duality" result for the equivariant K-theory of G/H where G and H satisfy the hypothesis of our Theorem.

The notation is borrowed from [3]. $R(G)$ is the *complex representation ring of* G, R denotes the *set of roots* with R^+ the subset of *positive roots* relative to some ordering of the Weyl chambers. If w is a weight (e.g. a root) we let e^w denote the associated element of $R(T)$ where T is a *maximal torus* of G. $W(G)$ is the *Weyl group* of G, and we call an element $x \in R(T)$ alternating, if $\phi(x) = (\mathrm{sgn}\, \phi)x$ for all $\phi \in W(G)$.

The adjoint representation restricted to a maximal torus T provides a real representation $[g/t]$ whose complexification $[g/t] \otimes \mathbf{C}$ equals $\sum_{\alpha \in R^+}(e^\alpha + e^{-\alpha})$. Finally, we let

$$\lambda_{-1}(G/T) \equiv \sum (-1)^p \lambda^p [g/t] \otimes \mathbf{C}.$$

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By the multiplicative behavior of λ_t (see [2, p. 118], e.g.), or directly for that matter, we see that

$$\lambda_{-1}(G/T) = \prod_{\alpha \in R^+} (1 - e^{\alpha})(1 - e^{-\alpha}).$$

I would like to thank A. T. Vasquez for the present form of the proof. It is essentially the original done more elegantly.

Theorem. *For a compact connected Lie group G the following conditions are equivalent*:

(1) Ad: $G \to SO(n)$ *factors through* π: $\mathrm{Spin}^c(n) \to SO(n)$.

(2) *There exists a character* ρ: $G \to S^1$ *such that* $(\Sigma_{\alpha \in R^+} \alpha) + s = 2w$. *$s$ is the weight corresponding to ρ restricted to T and w is a weight of G.*

(3) *$R(T)$ contains an alternating element $\Omega(G)$ such that* $\Omega(G)\Omega(G)^* = \lambda_{-1}(G/T)$.

Before giving the proof, which is straightforward, we comment that for *semisimple* G's the only character of G is the trivial one, so that s must be 0. Condition (2) in that case is the more familiar one that "one-half the sum of the positive roots is a weight." In this case there is no need for $\mathrm{Spin}^c(n)$; we need only consider the more familiar 2-fold covering π: $\mathrm{Spin}(n) \to SO(n)$.

Proof. For $n \geq 3$ we have an isomorphism $\pi_1(SO(n) \times SO(2)) \cong Z/2Z \oplus Z$. Let θ: $\pi_1 \to Z/2Z$ be the homomorphism corresponding to mod 2 addition. By covering space theory there is a 2-fold covering space corresponding to the subgroup ker θ of π_1. It can be readily checked that p: $\mathrm{Spin}^c(n) \to SO(n) \times SO(2)$ is this covering, and that π: $\mathrm{Spin}^c(n) \to SO(n)$ is the composition of p and projection onto the first factor. Alternatively one can take this as a definition of $\mathrm{Spin}^c(n)$. Thus it follows from covering space theory that a homomorphism ϕ: $G \to SO(n)$ factors through π if and only if for some homomorphism ρ: $G - SO(2)$ the composition

$$\pi_1(G) \xrightarrow{\pi_1(\phi,\rho)} \pi_1(SO(n) \times SO(2)) \xrightarrow{\theta} Z/2Z$$

is zero.

Suppose now that $\mathrm{Ad} = \pi \circ \widetilde{\mathrm{Ad}}$ for some homomorphism $\widetilde{\mathrm{Ad}}$: $G \to \mathrm{Spin}^c(n)$; then $p \circ \widetilde{\mathrm{Ad}} = (\mathrm{Ad}, \rho)$ for some homomorphism ρ: $G \to SO(2)$. We may identify the integer lattice I with $\pi_1(T)$ [1, p. 129]. We know then that the composition

$$I \cong \pi_1(T) \to \pi_1(G) \to \pi_1(SO(n) \times SO(2)) \cong Z/2\,Z \oplus Z \xrightarrow{\theta} Z/2Z$$

is zero. A moment's thought (see [1, p. 136]) shows that this homomorphism is given by $\Sigma\alpha + s$ where the α's are the positive roots of G, and s is the weight corresponding to the restriction of ρ to T. Thus $(\Sigma\alpha + s)/2 = w$ is integer valued on I and hence is a weight. This shows, when $n \geq 3$, that (1) implies (2). The argument is clearly reversible (it is necessary to recall that $\pi_1(T) \to \pi_1(G)$ is onto). If $n \leq 2$ the Theorem is trivial, for G is abelian!

We now show that (2) implies (3). We will show that $\Omega(G) = e^{-w}\Pi(1 - e^{\alpha})$ will work. Let $s_{\alpha_1} \in W(G)$ denote "reflection in the wall $\alpha_1 = 0$". $W(G)$ is generated by such elements, thus to show $\Omega(G)$ is alternating is to show that $s_{\alpha_1}(\Omega(G)) = -\Omega(G)$. It is known (see [1, 5.39]) that $s_{\alpha_1}(\alpha_1) = -\alpha_1$ and that s_{α_1} permutes the other positive roots among themselves. Thus $s_{\alpha_1}(\Pi(1 - e^{\alpha})) = -e^{-\alpha_1}\Pi(1 - e^{\alpha})$. Since $e^s \in R(G) = R(T)^{W(G)}$, we have $s_{\alpha_1}(s) = s$. Thus $s_{\alpha_1}(w) = w - \alpha_1$. Hence $s_{\alpha_1}(\Omega(G)) = -\Omega(G)$. From the formula for $\lambda_{-1}(G/T)$ we see that $\Omega(G)\Omega(G)^* = \lambda_{-1}(G/T)$.

We turn now to the proof that (3) implies (2). By [1, p. 147], since $\Omega(G)$ is alternating, we can write $\Omega(G) = a\Pi(1 - e^{\alpha})$ for some $a \in R(T)$. If $\Omega(G)\Omega(G)^* = \lambda_{-1}(G/T)$ we deduce that $\lambda_{-1}(G/T) = aa^*\lambda_{-1}(G/T) \in R(T)$. Thus $1 = aa^*$ and a is a unit. A look at the structure of $R(T) \cong Z[x_1, \ldots, x_l, x_1^{-1}, \ldots, x_l^{-1}]$ shows that such an a corresponds to an element of the form $\pm x_1^{n_1} \cdots x_l^{n_l}$ (i.e. $a = \pm e^{-w}$ for some weight w). Thus $e^{-w}\Pi(1 - e^{\alpha})$ is aletrnating. Reversing one of the above calculations shows that $s_{\alpha_1}(w) = w - \alpha_1$. Hence $s = -\Sigma\alpha + 2w$ is fixed by s_α. Hence it is fixed by $W(G)$. Let $e^s\colon T \to S^1$ be the corresponding character. We wish to show that e^s is the restriction of a homomorphism $\rho\colon G \to S^1$. Suppose Ad factors through Spin(n). We may use [1, 6.19] and the notation introduced there. There must be an h such that $A(h) = \delta$. On the other hand $A(he^s) = \pm\delta\rho$ for some irreducible representation ρ of G. Since s is invariant, $A(he^s) = A(h)e^s = \delta e^s$. Thus $e^s = \pm\rho$. Viewing both as functions on T and evaluating at the identity, we see that the $+$ sign must hold (i.e. e^s is the restriction of ρ). If Ad does not lift to Spin we may find a 2-fold cover $f\colon \tilde{G} \to G$ for which Ad does lift. Applying the argument above we will find a homomorphism $\tilde{\rho}\colon \tilde{G} \to S^1$ which is $e^s \circ f$ on the inverse image of T. In particular, $\tilde{\rho}$ is trivial on the kernel of f; thus $\tilde{\rho}$ is of the form $\rho \circ f$ and this ρ is what we seek.

Definition. We say that G satisfies *condition* B if and only if it is a compact connected Lie group satisfying any (and hence all) of the above conditions.

Remarks. As noted by Bott [3], if $\pi_1(G)$ has no 2-torsion, G satisfies condition B. This is, of course, implicitly proved in the above argument. However, this is not a necessary condition. For example, $G = SO(2n)$ works. Indeed in this case $\frac{1}{2}\Sigma\alpha$ is a weight. Despite a remark to the contrary in [3], neither the π_1 condition nor condition B is inherited by subgroups H of maximal rank. For example, consider $G = \mathrm{Spin}(k + l)$ and H the inverse image of $SO(k) \times SO(l) \hookrightarrow SO(k + l)$ for k odd ≥ 3. Finally, condition B brings the Weyl character formula into play. For, substituting w for $\frac{1}{2}\Sigma_{\alpha\in R^+}\alpha$ in [1, 6.6, 6.16, 6.18] does not change the proof.

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