CONDITIONS ON A COMPACT CONNECTED LIE GROUP
WHICH INSURE A "WEYL CHARACTER FORMULA"

JACK M. SHAPIRO

ABSTRACT. A theorem showing the equivalence of three conditions
on a compact connected Lie group is proved. Among the corollaries is an
extended "Weyl character formula" as originally stated by Bott.

In [3] Bott states a generalized "Weyl character formula" for compact
connected Lie groups $G$ whose fundamental group $\pi_1(G)$ has no 2-torsion
(recall that the original formula is stated for $G$ simply connected). As in
the original character formula the critical issue surrounds the lifting of the
adjoint representation $\text{Ad}: G \to \text{SO}(n)$. For $\pi_1(G) = 0$ $\text{Ad}$ can be lifted to
$\text{Spin}(n)$ where $\pi: \text{Spin}(n) \to \text{SO}(n)$ is the usual double cover. In the case
where $\pi_1(G)$ has no 2-torsion, $\text{Ad}$ can be lifted to $\text{Spin}^c(n)$.

The purpose of this note is to prove that three properties of compact
connected Lie groups are equivalent. A corollary is a proof of the assertion
in [3, p. 178] that a character formula exists for any compact connected $G$
where $\pi_1(G)$ has no 2-torsion. The Theorem also lays the groundwork for
large number of special cases; a "Poincaré duality" result for the equivari-
ant $K$-theory of $G/H$ where $G$ and $H$ satisfy the hypothesis of our Theorem.

The notation is borrowed from [3]. $R(G)$ is the complex representation
ring of $G$, $R$ denotes the set of roots with $R^+$ the subset of positive roots
relative to some ordering of the Weyl chambers. If $w$ is a weight (e.g. a
root) we let $e^w$ denote the associated element of $R(T)$ where $T$ is a maxi-
mal torus of $G$. $W(G)$ is the Weyl group of $G$, and we call an element $x \in
R(T)$ alternating, if $\langle \phi, x \rangle = (\text{sgn } \phi)x$ for all $\phi \in W(G)$.

The adjoint representation restricted to a maximal torus $T$ provides a
real representation $[g/\mathfrak{t}]$ whose complexification $[g/\mathfrak{t}] \otimes \mathbb{C}$ equals
$\sum_{a \in R^+} \langle e^a + e^{-a} \rangle$. Finally, we let
$$\lambda_{-1}(G/T) = \sum (-1)^p \lambda^p[g/\mathfrak{t}] \otimes \mathbb{C}.$$
By the multiplicative behavior of $\lambda_t$ (see [2, p. 118], e.g.), or directly for that matter, we see that

$$\lambda_{-1}(G/T) = \prod_{a \in \mathbb{R}^+} (1 - e^a)(1 - e^{-a}).$$

I would like to thank A. T. Vasquez for the present form of the proof. It is essentially the original done more elegantly.

**Theorem.** For a compact connected Lie group $G$ the following conditions are equivalent:

1. Ad: $G \to SO(n)$ factors through $\pi$: Spin$^c(n) \to SO(n)$.
2. There exists a character $\rho$: $G \to \mathbb{S}^1$ such that $(\Sigma_{a \in \mathbb{R}^+} + s) = 2w$, $s$ is the weight corresponding to $\rho$ restricted to $T$ and $w$ is a weight of $G$.
3. $R(T)$ contains an alternating element $\Omega(G)$ such that $\Omega(G) \Omega(G)^* = \lambda_{-1}(G/T)$.

Before giving the proof, which is straightforward, we comment that for semisimple $G$'s the only character of $G$ is the trivial one, so that $s$ must be 0. Condition (2) in that case is the more familiar one that "one-half the sum of the positive roots is a weight." In this case there is no need for Spin$^c(n)$; we need only consider the more familiar 2-fold covering $\pi$: Spin$^c(n) \to SO(n)$.

**Proof.** For $n \geq 3$ we have an isomorphism $\pi_1(SO(n) \times SO(2)) \cong \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}$. Let $\theta: \pi_1 \to \mathbb{Z}/2\mathbb{Z}$ be the homomorphism corresponding to mod 2 addition. By covering space theory there is a 2-fold covering space corresponding to the subgroup ker $\theta$ of $\pi_1$. It can be readily checked that $p: Spin^c(n) \to SO(n) \times SO(2)$ is this covering, and that $\pi: Spin^c(n) \to SO(n)$ is the composition of $p$ and projection onto the first factor. Alternatively one can take this as a definition of Spin$^c(n)$. Thus it follows from covering space theory that a homomorphism $\phi$: $G \to SO(n)$ factors through $\pi$ if and only if for some homomorphism $\rho$: $G \to SO(2)$ the composition

$$\pi_1(G) \xrightarrow{\pi_1(\phi, \rho)} \pi_1(SO(n) \times SO(2)) \xrightarrow{\theta} \mathbb{Z}/2\mathbb{Z}$$

is zero.

Suppose now that $Ad = \pi \circ \hat{Ad}$ for some homomorphism $\hat{Ad}$: $G \to Spin^c(n)$; then $p \circ \hat{Ad} = (Ad, \rho)$ for some homomorphism $\rho$: $G \to SO(2)$. We may identify the integer lattice $I$ with $\pi_1(T)$ [1, p. 129]. We know then that the composition

$$I \cong \pi_1(T) \to \pi_1(G) \to \pi_1(SO(n) \times SO(2)) \cong \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z} \xrightarrow{\theta} \mathbb{Z}/2\mathbb{Z}$$
is zero. A moment's thought (see [1, p. 136]) shows that this homomorphism is given by \( \Sigma a + s \) where the \( a \)'s are the positive roots of \( G \), and \( s \) is the weight corresponding to the restriction of \( \rho \) to \( T \). Thus \( \frac{\Sigma a + s}{2} = w \) is integer valued on \( l \) and hence is a weight. This shows, when \( n \geq 3 \), that (1) implies (2). The argument is clearly reversible (it is necessary to recall that \( \pi_1(T) \rightarrow \pi_1(G) \) is onto). If \( n \leq 2 \) the Theorem is trivial, for \( G \) is abelian!

We now show that (2) implies (3). We will show that \( \Omega(G) = e^{-w} \Pi(1 - e^a) \) will work. Let \( s_{a_1} \in W(G) \) denote "reflection in the wall \( a_1 = 0''). W(G) is generated by such elements, thus to show \( \Omega(G) \) is alternating is to show that \( s_{a_1}(\Omega(G)) = -\Omega(G) \). It is known (see [1, 5.39]) that \( s_{a_1}(a_1) = -a_1 \) and that \( s_{a_1} \) permutes the other positive roots among themselves. Thus \( s_{a_1}(\Pi(1 - e^a)) = -e^{-a_1} \Pi(1 - e^a) \). Since \( e^s \in R(G) = R(T)^W(G) \), we have \( s_{a_1}(s) = s \). Thus \( s_{a_1}(w) = w - a_1 \). Hence \( s_{a_1}(\Omega(G)) = -\Omega(G) \). From the formula for \( \lambda_{-1}(G/T) \) we see that \( \Omega(G)\Omega(G)^* = \lambda_{-1}(G/T) \).

We turn now to the proof that (3) implies (2). By [1, p. 147], since \( \Omega(G) \) is alternating, we can write \( \Omega(G) = a \Pi(1 - e^a) \) for some \( a \in R(T) \). If \( \Omega(G)\Omega(G)^* = \lambda_{-1}(G/T) \) we deduce that \( \lambda_{-1}(G/T) = a a^* \lambda_{-1}(G/T) \in R(T) \). Thus \( 1 = a a^* \) and \( a \) is a unit. A look at the structure of \( R(T) \cong \mathbb{Z}[x_1, \ldots, x_l, x_1^{-1}, \ldots, x_l^{-1}] \) shows that such an \( a \) corresponds to an element of the form \( \pm x_1^n \cdots x_l^n \) (i.e. \( a = \pm e^{-w} \) for some weight \( w \)). Thus \( e^{-w} \Pi(1 - e^a) \) is alternating. Reversing one of the above calculations shows that \( s_{a_1}(w) = w - a_1 \). Hence \( s = -\Sigma a + 2w \) is fixed by \( s_{a_1} \). Hence it is fixed by \( W(G) \). Let \( e^s \): \( T \rightarrow S^1 \) be the corresponding character. We wish to show that \( e^s \) is the restriction of a homomorphism \( \rho \): \( G \rightarrow S^1 \). Suppose \( Ad \) factors through \( \text{Spin}(n) \). We may use [1, 6.19] and the notation introduced there. There must be an \( h \) such that \( A(h) = \delta \). On the other hand \( A(h e^s) = \pm \delta \rho \) for some irreducible representation \( \rho \) of \( G \). Since \( s \) is invariant, \( A(h e^s) = A(h)e^s = \delta e^s \). Thus \( e^s = \pm \rho \). Viewing both as functions on \( T \) and evaluating at the identity, we see that the + sign must hold (i.e. \( e^s \) is the restriction of \( \rho \)). If \( Ad \) does not lift to \( \text{Spin} \) we may find a 2-fold cover \( f \): \( \tilde{G} \rightarrow G \) for which \( Ad \) does lift. Applying the argument above we will find a homomorphism \( \tilde{\rho} \): \( \tilde{G} \rightarrow S^1 \) which is \( e^s \circ f \) on the inverse image of \( T \). In particular, \( \tilde{\rho} \) is trivial on the kernel of \( f \); thus \( \tilde{\rho} \) is of the form \( \rho \circ f \) and this \( \rho \) is what we seek.

Definition. We say that \( G \) satisfies condition B if and only if it is a compact connected Lie group satisfying any (and hence all) of the above conditions.
Remarks. As noted by Bott [3], if $\pi_1(G)$ has no 2-torsion, $G$ satisfies condition B. This is, of course, implicitly proved in the above argument. However, this is not a necessary condition. For example, $G = SO(2n)$ works. Indeed in this case $\frac{1}{2}\Sigma \alpha$ is a weight. Despite a remark to the contrary in [3], neither the $\pi_1$ condition nor condition B is inherited by subgroups $H$ of maximal rank. For example, consider $G = \text{Spin}(k + l)$ and $H$ the inverse image of $SO(k) \times SO(l) \hookrightarrow SO(k + l)$ for $k$ odd $\geq 3$. Finally, condition B brings the Weyl character formula into play. For, substituting $w$ for $\frac{1}{2}\Sigma_{\alpha \in \mathbb{R}^+} \alpha$ in [1, 6.6, 6.16, 6.18] does not change the proof.

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130

INSTITUTE OF MATHEMATICS TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL