EVERYWHERE DIFFERENTIABLE FUNCTIONS
AND THE DENSITY TOPOLOGY

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Katznelson and Stromberg [3] have recently given a relatively simple proof of the existence of an everywhere differentiable function which is not monotone in any interval. Our purpose is to show the connection between this property and the density topology.

The density topology for the reals is completely regular [1], [2]. Since countable sets are closed in this topology, for each countable $S$ and $\xi \notin S$ there is an approximately continuous $f$ such that $0 \leq f(x) \leq 1$, $f(\xi) = 1$ and $f(x) = 0$ for each $x \in S$. Let $A$ and $B$ be disjoint countable sets each dense in the reals. They may be enumerated $A = \{a_n\}$, $B = \{b_n\}$. For each $n$, let $f_n$ be approximately continuous, $0 \leq f_n(x) \leq 1$, $f_n(a_n) = 1$, and $f_n(x) = 0$ for each $x \in B$, and let $g_n$ be approximately continuous, $0 \leq g_n(x) \leq 1$, $g_n(b_n) = 1$, and $g_n(x) = 0$ for each $x \in A$. The function

$$f = \sum_{n=1}^{\infty} 2^{-n} f_n - \sum_{n=1}^{\infty} 2^{-n} g_n$$

is bounded, approximately continuous, positive on $A$ and negative on $B$. Let $F$ be an indefinite integral of $f$. Then $F$ is everywhere differentiable and $F' = f$. So $F$ is not monotonic in any interval.

REFERENCES