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ANALYTIC DOMINATION BY FRACTIONAL POWERS WITH LINEAR ESTIMATES

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ABSTRACT. Conditions are given which imply the analytic domination of one operator by fractional powers of a positive selfadjoint operator. The conditions involve only linear estimates rather than the usual quadratic estimates.

Let $(\mathcal{H}, \|\cdot\|)$ be a normed linear space. For any linear operator A on $\mathcal{H}$, let $D(A)$ denote the domain of A and let $D^\infty(A) = \bigcap_{n=0}^\infty D(A^n)$. An element v in $D^\infty(A)$ is called an *analytic vector* for A if there is an $s > 0$ such that $\sum_{n=0}^\infty (s^n/n!)\|A^n v\| < \infty$. The set of all analytic vectors for A will be written $D^\omega(A)$. Let X denote another linear operator on $\mathcal{H}$. A *analytically dominates* X if every analytic vector for A is also an analytic vector for X, or equivalently, if $D^\omega(A) \subset D^\omega(X)$.

The following theorem gives conditions which guarantee analytic domination of X by $A^{1/k}$, k a positive integer, and can be found in the Appendix of [1].

Theorem [Nelson]. *Let A and X be everywhere defined linear operators on a normed linear space $\mathcal{H}$ such that*

$$\|X^r u\| \le \|Au\|, \qquad r = 1, 2, \ldots, k, \tag{1}$$

and

$$\|(\operatorname{ad} X)^n(A)u\| \le n!\|Au\|, \tag{2}$$

$n = 1, 2, \ldots,$ *and* $u \in \mathcal{H}$.

If $v \in D^\infty(A)$ satisfies

$$\|A^n v\| \le M^n(kn)! \tag{3}$$

for some $M > 0$ then v is an analytic vector for X.

Here $(\operatorname{ad} X)(A) = XA - AX$ *while for* $n \ge 1$,

$$(\operatorname{ad} X)^{n+1}(A) = (\operatorname{ad} X)((\operatorname{ad} X)^n(A)).$$

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In [2] the case $k = 1$ of the above theorem was investigated with $A \geq CI > 0$ being a selfadjoint operator on the Hilbert space $\mathcal{H}$. It was found that the quadratic estimates (1) and (2) of the type $\pm T^2 \leq A^2$ could be replaced by linear estimates $\pm T \leq A$ in the presence of some (skew-) symmetry. In this paper similar results are obtained for $k = 2, 3, \ldots$. Specifically we will prove the following result.

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space $\mathcal{H}$ with* $\inf(\text{spectrum}(A)) > 0$. *Let $X: D^\infty(A) \to D^\infty(A)$. Assume*

$$\|A^{-1/2}X^kA^{-1/2}u\| \leq \|u\| \tag{4}$$

for $k = 1, 2, \ldots, r$ and $u \in D^\infty(A)$ and

$$\|A^{-1/2}((\mathrm{ad}\, X)^n(A))A^{-1/2}u\| \leq n!\|u\| \tag{5}$$

for $n = 1, 2, \cdots$ and $u \in D^\infty(A)$.

If $v \in D^\infty(A)$ satisfies

$$\|A^nv\| \leq M^n(rn)! \tag{6}$$

for some M and $n = 1, 2, \ldots$, then v is an analytic vector for X.

This theorem is proved with the procedure developed in [2]. First observe that if v satisfies (6) then so does Av. Next renorm $\mathcal{H}$ with $|||\cdot|||$ defined by $|||u||| = \|A^{-1/2}u\|$ and use Nelson's theorem to show that Av is an analytic vector for X relative to $|||\ |||$. Then we appeal to a result from [2] and argue that if w is an analytic vector for X relative to $|||\ |||$ then $A^{-1}w$ is an analytic vector for X relative to $\|\ \|$. Finally we apply the last argument to Av and conclude that $v = A^{-1}(Av) \in D^\omega(X)$.

In more detail, we have the

Proof of Theorem 1. Observe that any v satisfying (6) also satisfies

$$\sum_{n=0}^{\infty} \frac{((2M)^{-1/r})^n}{n!}\|(A^{1/r})^nv\| \leq \sum_{k=0}^{\infty} \frac{((2M)^{-1/r})^{k\cdot r}}{(kr)!}\|(A^{1/r})^{k\cdot r}v\| \leq \sum_{k=0}^{\infty} 2^{-k}$$

so that $v \in D^\omega(A^{1/r})$. Conversely, if $v \in D^\omega(A^{1/r})$ then for some $t > 0$, $\sum_{n=0}^\infty (t^n/n!)\|A^{n/r}v\| < \infty$ and so there is an $M < \infty$ such that $\|A^{n/r}v\| \leq n!M^n$ and $\|A^kv\| \leq (M^r)^k(nr)!$. Thus the set of all v's satisfying (6) for varying M's is exactly $D^\omega(A^{1/r})$. Consequently if v satisfies (6) there is a $t > 0$ such that $v = \exp(-tA^{1/r})w$ for some $w \in \mathcal{H}$. Since $Av = \exp(-\frac{t}{2}A^{1/r})A\exp(-\frac{t}{2}A^{1/r})w$, Av is also in $D^\omega(A^{1/r})$ and so satisfies (6). That is, if v satisfies (6) then there is a $Q < \infty$ such that

$$\|A^n(Av)\| \leq Q^n(r \cdot n)!. \tag{7}$$

Now observe that

$$|||X^k u||| = \|A^{-1/2} X^k u\| = \|A^{-1/2} X^k A^{-1/2} A^{1/2} u\| \leq \|A^{1/2} u\|$$

and so

$$|||X^k u||| \leq |||Au||| \tag{8}$$

for $k = 1, 2, \ldots, r$ and $u \in D^\infty(A)$.

Similarly

$$|||(\mathrm{ad}\ X)^n(A)u||| \leq n!|||Au||| \tag{9}$$

for $n = 1, 2 \ldots,$ and all u in $D^\infty(A)$.

With no loss of generality we may assume $\inf(\text{spectrum}(A)) \geq 1$. Then $|||u||| \leq \|u\|$ and any v satisfying (7) will also satisfy

$$|||A^n(Av)||| \leq Q^n(r \cdot n)!. \tag{10}$$

With (8), (9) and (10) Nelson's theorem may be applied to the space $D^\infty(A)$ normed with $|||\cdot|||$ yielding an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|A^{-1/2} X^n A v\| < \infty. \tag{11}$$

Combining the hypothesis of the theorem with Lemmas 3 and 4 of [2] permits the conclusion that $v = A^{-1}(Av)$ is an analytic vector for X relative to $\|\ \|$. Q.E.D.

Corollary. *$A^{1/r}$ analytically dominates X.*

Remark. If X is symmetric or skew-symmetric then (4) and (5) are equivalent to $|(X^k u, u)| \leq (Au, u)$ and $|((\mathrm{ad}\ X)^n(A)u, u)| \leq (Au, u)$.

REFERENCES

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