PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 1, August 1975

UNITARY HARMONIC NUMBERS

PETER HAGIS, JR. AND GRAHAM LORD

ABSTRACT. If $d^*(n)$ and $\sigma^*(n)$ denote the number and sum, respectively, of the unitary divisors of the natural number n then the harmonic mean of the unitary divisors of n is given by $H^*(n) = nd^*(n)/\sigma^*(n)$. Here we investigate the properties of $H^*(n)$, and, in particular, study those numbers n for which $H^*(n)$ is an integer.

1. **Introduction.** Let $d(n)$ and $\sigma(n)$ denote, respectively, the number and sum of the positive divisors of the natural number n. Ore [6] showed that the harmonic mean of the positive divisors of n is given by $H(n) = nd(n)/\sigma(n)$, and several papers (see [1], [5], [6], [7]) have been devoted to the study of $H(n)$. In particular the set of numbers S for which $H(n)$ is an integer has attracted the attention of number theorists, since the set of perfect numbers is a subset of S. The elements of S are called *harmonic numbers* by Pomerance [7]. This paper is devoted to a study of the unitary analogue of $H(n)$. We recall that the positive integer d is said to be a unitary divisor of n if $d|n$ and $(d, n/d) = 1$. It is easy to verify that if the canonical prime decomposition of n is given by

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \tag{1}$$

and $d^*(n)$ and $\sigma^*(n)$ denote the number and sum, respectively, of the unitary divisors of n then

$$d^*(n) = 2^k; \qquad \sigma^*(n) = (p_1^{\alpha_1} + 1)(p_2^{\alpha_2} + 1) \cdots (p_k^{\alpha_k} + 1). \tag{2}$$

It is also easy to show that the unitary harmonic mean (the harmonic mean of the unitary divisors) of n is given by

$$H^*(n) = nd^*(n)/\sigma^*(n) = \prod_{i=1}^{k} 2p_i^{\alpha_i}/(p_i^{\alpha_i} + 1). \tag{3}$$

We shall say that n is a *unitary harmonic number* if $H^*(n)$ is an integer,

---

Presented to the Society, January 25, 1975; received by the editors April 29, 1974.
AMS (*MOS*) *subject classifications* (1970). Primary 10A20.

and shall denote by UH the set of these numbers. A computer search (which required approximately 2.5 hours of CDC 6400 time at the Temple University Computer Center) was made for the elements of UH in the interval $[1, 10^6]$, and 45 such numbers were found. These are given in Table I at the end of this paper.

Subbarao and Warren [10] have defined n to be a *unitary perfect number* if $\sigma^*(n) = 2n$. Five such numbers are presently known [9]. Since $d^*(n)$ is even the following result is immediate from (3).

Proposition 1. *The set of unitary perfect numbers is a subset of* UH.

2. **Some elementary results concerning $H^*(n)$ and UH.** We now establish some facts which will be of use in the sequel and which, in certain cases, are of some interest in themselves. n will always denote a natural number with prime decomposition as given in (1); p, q, r with or without subscripts will always denote primes.

Lemma 1. $2^{k+1}/(k+2) \le H^*(n) < 2^k$ *with equality on the left if and only if* $n = 2$ *or* 6.

Proof. Since $x/(x+1)$ is monotonic increasing and bounded by 1 for positive x, it follows from (3) that

$$2^k > H^*(n) \ge 2^k(2/3)(3/4)(4/5)\cdots((k+1)/(k+2)) = 2^{k+1}/(k+2).$$

Lemma 2. *If* $p^\alpha \| n$, *then* $p^\alpha \ge H^*(n)/(2^k - H^*(n))$ *with equality if and only if* $k = 1$.

Proof. From (3), $H^*(n) \le 2^k p^\alpha/(p^\alpha + 1)$.

Lemma 3. *If* p^a $\{r^c\}$ *is the minimum* $\{$*maximum*$\}$ *prime power divisor of* n *in* (1) *then*

$$p^a \le kH^*(n)/(2^k - H^*(n)) \qquad \{r^c \ge ((k-1)2^k + H^*(n))/(2^k - H^*(n))\}$$

with equality if and only if $k = 1$ *or* $n = p^a q^b r^c$ *where* $q^b = p^a + 1$ (*so that* $2|n$) *and* $r^c = p^a + 2$ *or* $c = 0$ $\{$*where* $q^b = r^c - 1$ (*so that* $2|n$) *and* $p^a = r^c - 2$ *or* $a = 0\}$.

Proof.

$$H^*(n) \ge 2^k\{p^a/(p^a+1)\}\{(p^a+1)/(p^a+2)\}\cdots\{(p^a+k-1)/(p^a+k)\}$$

$$= 2^k p^a/(p^a + k),$$

and

$$H^*(n) \leq 2^k\{(r^c - k + 1)/(r^c - k + 2)\}\{(r^c - k + 2)/(r^c - k + 3)\} \cdots \{r^c/(r^c + 1)\}$$

$$= 2^k(r^c - k + 1)/(r^c + 1)$$

with equality only in the specified "exceptional" cases.

We turn now to some results concerning UH. The following proposition was proved by Ore [6, p. 617] for harmonic numbers and holds for elements of UH since $H(n) = H^*(n)$ if and only if n is square-free.

Proposition 2. *If n is square-free and $n \neq 6$, then n is not a unitary harmonic number.*

Since $2|(p^\alpha + 1)$ if p is odd, and since $4|(p^\alpha + 1)$ if $p = 4j + 3$ and α is odd, our next two results follow immediately from (3).

Proposition 3. *If n is odd and $n \in UH$, then $H^*(n)$ is odd.*

Proposition 4. *If n is odd, $n \in UH$, $p^\alpha \| n$, and $p = 4j + 3$ then α is even.*

From (2) and (3) it is immediate that $H^*(n)$ is a multiplicative function. Therefore, if $(n, m) = 1$ then $H^*(nm) = H^*(n) \cdot md^*(m)/\sigma^*(m)$ from which we easily deduce the following result.

Proposition 5. *If $n \in UH$, $(p, n) = 1$, and $(p^\alpha + 1)|2H^*(n)$, then $p^\alpha n \in UH$.*

For example, since $40950 \in UH$ and $30 = 2H^*(40950)$ we see that $29 \cdot 40950 \in UH$ also.

3. Two cardinality theorems.

Theorem 1. *If S_c is the set of natural numbers n such that $H^*(n) = c$, then S_c is finite (or empty) for every real number c.*

Proof. Our proof is based on an idea due to Shapiro [8]. Since $2^{k+1}/(k + 2) \geq k$ we note first that if $H^*(n) = c$ then Lemma 1 implies that the number of prime factors of n is bounded (by c). Now assume that S_c is infinite. Then S_c must contain an *infinite* subset, say S_{cm}, each of whose elements has *exactly* m prime factors. It is not difficult to see that an infinite sequence $n_1, n_2, \cdots$ of distinct integers exists with the following properties:

(i) $n_i \in S_{cm}$ so that $H^*(n_i) = c$ for $i = 1, 2, \cdots$;

(ii) $$n_i = p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_{is}^{\alpha_{is}} \cdots p_{im}^{\alpha_{im}} = P \prod_{j=s}^{m} p_{ij}^{\alpha_{ij}}$$

where

$$p_1^{\alpha_1} < \cdots < p_{s-1}^{\alpha_{s-1}} < p_{is}^{\alpha_{is}} < \cdots < p_{im}^{\alpha_{im}} \quad \text{for } i = 1, 2, \cdots.$$

(P may be empty, but $s - 1 \neq m$.)

(iii) $p_{ij}^{\alpha_{ij}} \to \infty$ as $i \to \infty$ for $j = s, \cdots, m$.

(That is, each n_i is composed of a fixed, constant block of prime powers and a variable block of prime powers arranged monotonically within the block and such that each component of this variable block goes to infinity with i.)

From (i) and (ii) we see that

$$\frac{c}{H^*(P)} = \prod_{j=s}^{m} H^*(p_{ij}^{\alpha_{ij}}) < 2^{m+1-s}$$

so that there exists a fixed positive number v such that $\prod_{j=s}^{m} H^*(p_{ij}^{\alpha_{ij}}) = 2^{m+1-s} - v$ for $i = 1, 2, 3, \cdots$. But from (iii) it follows that $H^*(p_{ij}^{\alpha_{ij}}) \to 2$ as $i \to \infty$ for $s \leq j \leq m$. Therefore, for "large" i, $\prod_{j=s}^{m} H^*(p_{ij}^{\alpha_{ij}}) > 2^{m+1-s} - v$. This contradiction completes the proof.

Since there are only finitely many integers between $2^{k+1}/(k+2)$ and 2^k the following theorem follows from Lemma 1 and Theorem 1.

Theorem 2. *There exist at most finitely many unitary harmonic numbers with a specified number of distinct prime factors.*

From Proposition 1 we have the following corollary which was first proved by Subbarao and Warren [10].

Corollary 2.1. *There are at most a finite number of unitary perfect numbers with a specified number of prime factors.*

4. Elements of UH with a specified number of prime factors.

Let T_k denote the set of unitary harmonic numbers which have exactly k prime factors. In connection with Theorem 2 it is perhaps of some interest to identify the elements of T_k for a few selected values of k.

Proposition 6. *T_1 is empty.*

Proof. $H^*(p^\alpha) = 2p^\alpha/(p^\alpha + 1)$, and it is easy to see that $(p^\alpha + 1) \nmid 2p^\alpha$.

Proposition 7. $T_2 = \{6, 45\}$.

Proof. If $n \in T_2$ then, from Lemma 1, $H^*(n) = 2$ or 3. If $H^*(n) = 2$ then from Lemma 3, $2\|n$ and consequently $n = 6$. If $H^*(n) = 3$ then from (3) and Lemma 2, $3^b\|n$ where $b > 1$. From Lemmas 2 and 3 either $2^2\|n$ or $5\|n$. If $n = 2^2 3^b$ then $5(3^b + 1) = 16 \cdot 3^{b-1}$ which is impossible. If $n = 5 \cdot 3^b$ then $3 = H^*(n) \geq H^*(5 \cdot 3^2) = 3$ so that $n = 45$.

Proposition 8. $T_3 = \{60, 90, 1512, 15925, 55125\}$.

Proof. From Lemma 1 $H^*(n) = 4, 5, 6$ or 7 if $n \in T_3$. We consider these possibilities separately. p^a will always denote the minimal prime power in (1). It can be bounded by using Lemmas 2 and 3. We shall also rely heavily on the fact that $x/(x + 1)$ is monotonic increasing.

Case I. $H^*(n) = 4$. Then $p^a = 2$ or 3, and from Proposition 3 n is even. If $p^a = 2$, then, since n is not square-free, $4 = H^*(n) \geq H^*(2 \cdot 3^2 \cdot 5) = 4$. If $p^a = 3$ then $4 = H^*(n) \geq H^*(2^2 \cdot 3 \cdot 5) = 4$. Therefore, $n = 60$ or 90.

Case II. $H^*(n) = 5$. $p^a = 2, 3$ or 4, and from (3) $5|n$. If $p^a = 2$ then $n = 2 \cdot 3^b 5^c$, and from (3): $(3^{b-1} - 5)(5^{c-1} - 3) = 16$ which is impossible. If $p^a = 3$ then from Proposition 4 n is even and $n = 2^b 3 \cdot 5^c$. It follows that $(2^b - 5)(5^{c-1} - 1) = 6$ which is impossible. If $p^a = 4$ then $n = 4 \cdot 5^b q^c$ and $5 = H^*(n) \geq H^*(20q^c) = 16q^c/3(q^c + 1)$ which implies that $q^c = 7, 9, 11$ or 13. Each of these possibilities leads to a contradiction.

Case III. $H^*(n) = 6$. Then n is even, $3|n$, and $p^a = 4, 5, 7$ or 8. If $p^a = 4$ then $n = 4 \cdot 3^b \cdot 5^c$; if $p^a = 5$ then $n = 2^b 3^c 5$; if $p^a = 8$ then $n = 8 \cdot 3^b \cdot q^c$ where $q^c \geq 11$. From (3): $(3^{b-1} - 5)(5^{c-1} - 3) = 16$; or $(2^b - 9)(3^{c-2} - 1) = 10$; or $(5 \cdot 3^{b-3} - 1)(5q^c - 27) = 32$. None of these is possible. If $p^a = 7$ then $n = 2^b 3^c 7$, and $(2^{b-1} - 3)(3^{c-1} - 2) = 7$. Therefore, $b = c = 3$ and $n = 1512$.

Case IV. $H^*(n) = 7$. Then $7|n$ and $p^a = 8, 9, 11, 13, 16, 17$ or 19. Assume first that $2|n$. Then, if $p^a = 9, 11, 17$ or 19, n has *four* prime factors (from (3)). If $p^a = 13$ then $H^*(n) \geq H^*(2^5 7^3 13) > 7$; if $p^a = 16$ then $H^*(n) \geq H^*(16 \cdot 7^3 17) > 7$; if $p^a = 8$ then $n = 8 \cdot 3^b 7^c$, and $(3^{b-2} - 7) \cdot (7^{c-1} - 9) = 64$ which is impossible. Now assume that n is odd. Then, using Proposition 4, $7^2|n$; and $p^a = 9, 13$ or 17. If $p^a = 17$ then $H^*(n) \geq H^*(17 \cdot 3^4 \cdot 7^2) > 7$. If $p^a = 9$ then $n = 9 \cdot 5^b 7^c$, and $(5^{b-1} - 7)(7^{c-1} - 5) = 36$. Therefore, $b = 3$, $c = 2$ and $n = 55125$. If $p^a = 13$ then $n = 13 \cdot 7^b q^c$. If $b \geq 4$ then, from (3), $7^2|(q^c + 1)$ so that $q^c \geq 97$. But $H^*(13 \cdot 7^4 97) > 7$. Therefore, $b = 2$ and it then follows that $n = 13 \cdot 7^2 \cdot 5^2 = 15925$.

5. **The distribution of the unitary harmonic numbers.** For each positive real number x we shall denote by $A(x)$ the number of integers n such that

$n \leq x$ and $n \in UH$. This section is devoted to a proof of

Theorem 3. *For any* $\epsilon > 0$, $A(x) < 2.2x^{1/2}2^{(1+\epsilon)\log x/\log\log x}$ *for "large"* x.

Proof. We use an argument of Kanold [5]. A *powerful number* is a positive integer m with the property that if $p|m$ then $p^2|m$. It is obvious that every positive integer can be written uniquely in the form N_PN_F where $(N_P, N_F) = 1$, N_P is powerful (or 1) and N_F is square-free. If $P(x)$ is the number of powerful numbers not exceeding x it is proved in [2] that $P(x) \sim cx^{1/2}$ where $c = \zeta(3/2)/\zeta(3) = 2.173\cdots$. It follows that $P(x) < 2.2x^{1/2}$ for large x.

If N_P is a (fixed) powerful number let $g(N_P, x)$ denote the number of square-free numbers N_F such that $(N_P, N_F) = 1$, $N_PN_F \leq x$, and $N_PN_F \in UH$. If $G(x) = \max\{g(N_P, x)\}$ for $N_P \leq x$ it follows that

$$A(x) < 2.2x^{1/2}G(x) \quad \text{for large } x. \tag{4}$$

We now investigate the magnitude of $G(x)$. Let N_P be a powerful number for which square-free numbers $m_1, m_2, \cdots, m_{G(x)}$ exist such that $(N_P, m_i) = 1$, $N_Pm_i \leq x$ and $N_Pm_i \in UH$ for $i = 1, 2, \cdots, G(x)$. Then $H^*(N_Pm_i) = H^*(N_P)H^*(m_i) = Z_i$ where Z_i is an integer for $i = 1, \cdots, G(x)$. If $Z_i = Z_j$ where $i \neq j$, and $(m_i, m_j) = d$ then, of course, $H^*(M_i) = H^*(M_j)$ where $M_i = m_i/d$ and $M_j = m_j/d$. If $M_i = p_1 \cdots p_s$ (or 1) and $M_j = q_1 \cdots q_t$ (where $p_1 < \cdots < p_s$, $q_1 < \cdots < q_t$ and $p_u \neq q_v$) then from (3):

$$2^sp_1 \cdots p_s(1 + q_1) \cdots (1 + q_t) = 2^tq_1 \cdots q_t(1 + p_1) \cdots (1 + p_s). \tag{5}$$

It is not difficult to see that $p_s \leq 3$ and $q_t \leq 3$ so that (assuming $M_i < M_j$) $M_j = 6$, $M_i = 1$; $M_j = 3$, $M_i = 1$ or 2; $M_j = 2$, $M_i = 1$ are the only logical possibilities. Since none of these satisfies (5) we conclude that $Z_i \neq Z_j$ unless $i = j$. Therefore, without loss of generality, $Z_1 < Z_2 < \cdots < Z_{G(x)}$ so that $G(x) \leq Z_{G(x)} = H^*(N_Pm_{G(x)}) < 2^k$, where k is the number of prime factors in $N_Pm_{G(x)}$. If $N = 2 \cdot 3 \cdot 5 \cdots p_K$ is the "longest prime product" not exceeding x then $k \leq K$. Since $K \sim \log N/\log\log N$ (see [4, §22.10]) it follows that if $\epsilon > 0$ then

$$G(x) < 2^{(1+\epsilon)\log x/\log\log x} \quad \text{for large } x. \tag{6}$$

Our theorem follows from (4) and (6).

Remark. It follows easily from Theorem 3 that UH has zero density. That the set of unitary perfect numbers has density zero was first shown by Subbarao [3, p. 1117].

Table 1. The unitary harmonic numbers in $[1, 10^6]$.

n	$H^*(n)$	n	$H^*(n)$	n	$H^*(n)$
1	1	27300	15	232470	15
6	2	31500	10	257040	20
45	3	40950	15	330750	10
60	4	46494	9	332640	20
90	4	51408	12	464940	18
420	7	55125	7	565488	22
630	7	64260	17	598500	19
1512	6	66528	12	646425	13
3780	9	81900	18	661500	12
5460	13	87360	16	716625	13
7560	10	95550	14	790398	17
8190	13	143640	19	791700	29
9100	10	163800	20	859950	18
15925	7	172900	19	900900	33
16632	11	185976	12	929880	20

REFERENCES

1. M. Garcia, *On numbers with integral harmonic mean*, Amer. Math. Monthly 61 (1954), 89–96. MR 15, 506; 1140.

2. S. W. Golomb, *Powerful numbers*, Amer. Math. Monthly 77 (1970), 848–855. MR 42 #1780.

3. R. Guy and V. Klee, *Monthly research problems*, 1969–71, Amer. Math. Monthly 78 (1971), 1113–1122.

4. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

5. H.-J. Kanold, *Über das harmonische Mittel der Teiler einer natürlichen Zahl*, Math. Ann. 133 (1957), 371–374. MR 19, 635.

6. O. Ore, *On the averages of the divisors of a number*, Amer. Math. Monthly 55 (1948), 615–619. MR 10, 284.

7. C. Pomerance, *On a problem of Ore: harmonic numbers* (unpublished manuscript).

8. H. N. Shapiro, *Note on a theorem of Dickson*, Bull. Amer. Math. Soc. 55 (1949), 450–452. MR 10, 514.

9. M. V. Subbarao, *Are there an infinity of unitary perfect numbers?* Amer. Math. Monthly 77 (1970), 389–390.

10. M. V. Subbarao and L. J. Warren, *Unitary perfect numbers*, Canad. Math. Bull. 9 (1966), 147–153. MR 33 #3994.

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19121

SCHOOL OF BUSINESS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19121