ABSTRACT. If \( d^*(n) \) and \( \sigma^*(n) \) denote the number and sum, respectively, of the unitary divisors of the natural number \( n \) then the harmonic mean of the unitary divisors of \( n \) is given by \( H^*(n) = nd^*(n)/\sigma^*(n) \). Here we investigate the properties of \( H^*(n) \), and, in particular, study those numbers \( n \) for which \( H^*(n) \) is an integer.

1. Introduction. Let \( d(n) \) and \( \sigma(n) \) denote, respectively, the number and sum of the positive divisors of the natural number \( n \). Ore [6] showed that the harmonic mean of the positive divisors of \( n \) is given by \( H(n) = nd(n)/\sigma(n) \), and several papers (see [1], [5], [6], [7]) have been devoted to the study of \( H(n) \). In particular the set of numbers \( S \) for which \( H(n) \) is an integer has attracted the attention of number theorists, since the set of perfect numbers is a subset of \( S \). The elements of \( S \) are called harmonic numbers by Pomerance [7]. This paper is devoted to a study of the unitary analogue of \( H(n) \). We recall that the positive integer \( d \) is said to be a unitary divisor of \( n \) if \( d \mid n \) and \( (d, n/d) = 1 \). It is easy to verify that if the canonical prime decomposition of \( n \) is given by

\[
n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}
\]

and \( d^*(n) \) and \( \sigma^*(n) \) denote the number and sum, respectively, of the unitary divisors of \( n \) then

\[
d^*(n) = 2^k; \quad \sigma^*(n) = (p_1^{a_1} + 1)(p_2^{a_2} + 1) \cdots (p_k^{a_k} + 1).
\]

It is also easy to show that the unitary harmonic mean (the harmonic mean of the unitary divisors) of \( n \) is given by

\[
H^*(n) = nd^*(n)/\sigma^*(n) = \prod_{i=1}^{k} 2p_i^{a_i}/(p_i^{a_i} + 1).
\]

We shall say that \( n \) is a unitary harmonic number if \( H^*(n) \) is an integer.
and shall denote by $UH$ the set of these numbers. A computer search (which required approximately 2.5 hours of CDC 6400 time at the Temple University Computer Center) was made for the elements of $UH$ in the interval $[1, 10^6]$, and 45 such numbers were found. These are given in Table I at the end of this paper.

Subbarao and Warren [10] have defined $n$ to be a unitary perfect number if $\sigma^*(n) = 2n$. Five such numbers are presently known [9]. Since $d^*(n)$ is even the following result is immediate from (3).

**Proposition 1.** The set of unitary perfect numbers is a subset of $UH$.

2. Some elementary results concerning $H^*(n)$ and $UH$. We now establish some facts which will be of use in the sequel and which, in certain cases, are of some interest in themselves. $n$ will always denote a natural number with prime decomposition as given in (1); $p$, $q$, $r$ with or without subscripts will always denote primes.

**Lemma 1.** $2^{k+1}/(k + 2) \leq H^*(n) < 2^k$ with equality on the left if and only if $n = 2$ or 6.

**Proof.** Since $x/(x + 1)$ is monotonic increasing and bounded by 1 for positive $x$, it follows from (3) that

$$2^k > H^*(n) \geq 2^k(2/3)(3/4)(4/5) \cdots ((k + 1)/(k + 2)) = 2^{k+1}/(k + 2).$$

**Lemma 2.** If $p^a \| n$, then $p^a > H^*(n)/(2^k - H^*(n))$ with equality if and only if $k = 1$.

**Proof.** From (3), $H^*(n) \leq 2^k p^a/(p^a + 1)$.

**Lemma 3.** If $p^a \{r^c\}$ is the minimum {maximum} prime power divisor of $n$ in (1) then

$$p^a \leq k H^*(n)/(2^k - H^*(n)) \quad \{r^c \geq ((k - 1)2^k + H^*(n))/(2^k - H^*(n))\}$$

with equality if and only if $k = 1$ or $n = p^a q^b r^c$ where $q^b = p^a + 1$ (so that $2\|n$) and $r^c = p^a + 2$ or $c = 0$ {where $q^b = r^c - 1$ (so that $2\|n$) and $p^a = r^c - 2$ or $a = 0$}.

**Proof.**

$$H^*(n) \geq 2^k p^a/(p^a + 1)\{(p^a + 1)/(p^a + 2)\} \cdots \{(p^a + k - 1)/(p^a + k)\} = 2^k p^a/(p^a + k),$$
and
\[ H^*(n) \leq 2^k \left( \frac{r^c - k + 1}{r^c - k + 2} \right) \left( \frac{r^c - k + 2}{r^c - k + 3} \right) \cdots \left( \frac{r^c}{r^c + 1} \right) \]
\[ = 2^k \frac{r^c - k + 1}{r^c + 1} \]
with equality only in the specified "exceptional" cases.

We turn now to some results concerning \( \text{UH} \). The following proposition was proved by Ore [6, p. 617] for harmonic numbers and holds for elements of \( \text{UH} \) since \( H(n) = H^*(n) \) if and only if \( n \) is square-free.

**Proposition 2.** If \( n \) is square-free and \( n \neq 6 \), then \( n \) is not a unitary harmonic number.

Since \( 2|(p^\alpha + 1) \) if \( p \) is odd, and since \( 4|(p^\alpha + 1) \) if \( p = 4j + 3 \) and \( \alpha \) is odd, our next two results follow immediately from (3).

**Proposition 3.** If \( n \) is odd and \( n \in \text{UH} \), then \( H^*(n) \) is odd.

**Proposition 4.** If \( n \) is odd, \( n \in \text{UH} \), \( p^\alpha || n \), and \( p = 4j + 3 \) then \( \alpha \) is even.

From (2) and (3) it is immediate that \( H^*(n) \) is a multiplicative function. Therefore, if \( (n, m) = 1 \) then \( H^*(nm) = H^*(n) \cdot m d^*(m)/d^*(m) \) from which we easily deduce the following result.

**Proposition 5.** If \( n \in \text{UH}, (p, n) = 1, \) and \( (p^\alpha + 1)|2 H^*(n) \), then \( p^\alpha n \in \text{UH} \).

For example, since \( 40950 \in \text{UH} \) and \( 30 = 2 H^*(40950) \) we see that \( 29 \cdot 40950 \in \text{UH} \) also.

### 3. Two cardinality theorems.

**Theorem 1.** If \( S_c \) is the set of natural numbers \( n \) such that \( H^*(n) = c \), then \( S_c \) is finite (or empty) for every real number \( c \).

**Proof.** Our proof is based on an idea due to Shapiro [8]. Since \( 2^{k+1}/(k + 2) \geq k \) we note first that if \( H^*(n) = c \) then Lemma 1 implies that the number of prime factors of \( n \) is bounded (by \( c \)). Now assume that \( S_c \) is infinite. Then \( S_c \) must contain an infinite subset, say \( S_{cm} \), each of whose elements has exactly \( m \) prime factors. It is not difficult to see that an infinite sequence \( n_1, n_2, \ldots \) of distinct integers exists with the following properties:
(i) \( n_i \in S_{cm} \) so that \( H^*(n_i) = c \) for \( i = 1, 2, \ldots \);

(ii) \( n_i = p_1^{a_1} \cdots p_s^{a_s-1} p_{i s}^{a_i s} \cdots p_{i m}^{a_i m} = P \prod_{j=s}^{m} p_{ij}^{a_{ij}} \)

where

\( p_1^{a_1} < \cdots < p_s^{a_s-1} < p_{i s}^{a_i s} < \cdots < p_{i m}^{a_i m} \) for \( i = 1, 2, \ldots \).

(\( P \) may be empty, but \( s - 1 \neq m \).)

(iii) \( p_{ij}^{a_{ij}} \to \infty \) as \( i \to \infty \) for \( j = s, \ldots, m \).

(That is, each \( n_i \) is composed of a fixed, constant block of prime powers and a variable block of prime powers arranged monotonically within the block and such that each component of this variable block goes to infinity with \( i \).)

From (i) and (ii) we see that

\[
\frac{c}{H^*(P)} = \prod_{j=s}^{m} H^*(p_{ij}^{a_{ij}}) < 2^{m+1-s}
\]

so that there exists a fixed positive number \( \nu \) such that \( \prod_{j=s}^{m} H^*(p_{ij}^{a_{ij}}) = 2^{m+1-s} - \nu \) for \( i = 1, 2, 3, \ldots \). But from (iii) it follows that

\[
H^*(p_{ij}^{a_{ij}}) \to 2 \quad \text{as} \quad i \to \infty \quad \text{for} \quad s \leq j \leq m.
\]

Therefore, for "large" \( i \),

\[
\prod_{j=s}^{m} H^*(p_{ij}^{a_{ij}}) > 2^{m+1-s} - \nu.
\]

This contradiction completes the proof.

Since there are only finitely many integers between \( 2^{k+1}/(k + 2) \) and \( 2^k \) the following theorem follows from Lemma 1 and Theorem 1.

**Theorem 2.** There exist at most finitely many unitary harmonic numbers with a specified number of distinct prime factors.

From Proposition 1 we have the following corollary which was first proved by Subbarao and Warren [10].

**Corollary 2.1.** There are at most a finite number of unitary perfect numbers with a specified number of prime factors.

4. Elements of \( UH \) with a specified number of prime factors. Let \( T_k \) denote the set of unitary harmonic numbers which have exactly \( k \) prime factors. In connection with Theorem 2 it is perhaps of some interest to identify the elements of \( T_k \) for a few selected values of \( k \).

**Proposition 6.** \( T_1 \) is empty.

**Proof.** \( H^*(p^a) = 2p^a/(p^a + 1) \), and it is easy to see that \( (p^a + 1) \notin 2p^a \).

**Proposition 7.** \( T_2 = \{6, 43\} \).
Proof. If \( n \in T_2 \) then, from Lemma 1, \( H^*(n) = 2 \) or 3. If \( H^*(n) = 2 \) then from Lemma 3, \( 2^b || n \) and consequently \( n = 6 \). If \( H^*(n) = 3 \) then from (3) and Lemma 2, \( 3^b || n \) where \( b > 1 \). From Lemmas 2 and 3 either \( 2^2 || n \) or \( 5 || n \). If \( n = 2^2 3^b \) then \( 5(3^b + 1) = 16 \cdot 3^{b-1} \) which is impossible. If \( n = 5 \cdot 3^b \) then \( 3 = H^*(n) \geq H^*(5 \cdot 3^2) = 3 \) so that \( n = 45 \).

Proposition 8. \( T_3 = \{60, 90, 1512, 15925, 55125\} \).

Proof. From Lemma 1 \( H^*(n) = 4, 5, 6 \) or 7 if \( n \in T_3 \). We consider these possibilities separately. \( p^a \) will always denote the minimal prime power in (1). It can be bounded by using Lemmas 2 and 3. We shall also rely heavily on the fact that \( x/(x + 1) \) is monotone increasing.

Case I. \( H^*(n) = 4 \). Then \( p^a = 2 \) or 3, and from Proposition 3 \( n \) is even. If \( p^a = 2 \), then, since \( n \) is not square-free, \( 4 = H^*(n) \geq H^*(2 \cdot 3^b \cdot 5) = 4 \). If \( p^a = 3 \) then \( 4 = H^*(n) \geq H^*(2^2 \cdot 3 \cdot 5) = 4 \). Therefore, \( n = 60 \) or 90.

Case II. \( H^*(n) = 5 \). \( p^a = 2, 3 \) or 4, and from (3) \( 5 || n \). If \( p^a = 2 \) then \( n = 2 \cdot 3^b 5^c \), and from (3): \( (3^{b-1} - 5)(5^{c-1} - 3) = 16 \) which is impossible. If \( p^a = 3 \) then from Proposition 4 \( n \) is even and \( n = 2^b 3^c \cdot 5^c \). It follows that \( (2^b - 5)(5^{c-1} - 1) = 6 \) which is impossible. If \( p^a = 4 \) then \( n = 4 \cdot 5^b q^c \) and \( 5 = H^*(n) \geq H^*(20q^c) = 16q^c/3(q^c + 1) \) which implies that \( q^c = 7, 9, 11 \) or 13. Each of these possibilities leads to a contradiction.

Case III. \( H^*(n) = 6 \). Then \( n \) is even, \( 3 || n \), and \( p^a = 4, 5, 7 \) or 8. If \( p^a = 4 \) then \( n = 4 \cdot 3^b \cdot 5^c \); if \( p^a = 5 \) then \( n = 2^b 3^c 5^c \); if \( p^a = 8 \) then \( n = 8 \cdot 2^b \cdot q^c \) where \( q^c \geq 11 \). From (3): \( (3^{b-1} - 5)(5^{c-1} - 3) = 16 \); or \( (2^b - 9)(3^{b-2} - 1) = 10 \); or \( (5 \cdot 3^{b-3} - 1)(5q^c - 27) = 32 \). None of these is possible. If \( p^a = 7 \) then \( n = 2^b 3^c 7 \), and \( (2^{b-1} - 3)(3^{c-1} - 2) = 7 \). Therefore, \( b = c = 3 \) and \( n = 55125 \).

Case IV. \( H^*(n) = 7 \). Then \( 7 || n \) and \( p^a = 8, 9, 11, 13, 16, 17 \) or 19. Assume first that \( 2 || n \). Then, if \( p^a = 9, 11, 17 \) or 19, \( n \) has four prime factors (from (3)). If \( p^a = 13 \) then \( H^*(n) \geq H^*(2^5 3^7 13) > 7 \); if \( p^a = 16 \) then \( H^*(n) \geq H^*(16 \cdot 7^3 17) > 7 \); if \( p^a = 8 \) then \( n = 8 \cdot 3^b 7^c \), and \( (3^{b-2} - 7) \cdot (7^{c-1} - 9) = 64 \) which is impossible. Now assume that \( n \) is odd. Then, using Proposition 4, \( 7^2 || n \); and \( p^a = 9, 13 \) or 17. If \( p^a = 17 \) then \( H^*(n) \geq H^*(17 \cdot 3^4 \cdot 7^2) > 7 \). If \( p^a = 9 \) then \( n = 9 \cdot 5^b 7^c \), and \( (5^{b-1} - 7)(7^{c-1} - 5) = 36 \). Therefore, \( b = 3, c = 2 \) and \( n = 55125 \). If \( p^a = 13 \) then \( n = 13 \cdot 7^b q^c \). If \( b \geq 4 \) then, from (3), \( 7^2 || (q^c + 1) \) so that \( q^c \geq 97 \). But \( H^*(13 \cdot 7^{497}) > 7 \). Therefore, \( b = 2 \) and it then follows that \( n = 13 \cdot 7^2 \cdot 5^2 = 15925 \).

5. The distribution of the unitary harmonic numbers. For each positive real number \( x \) we shall denote by \( A(x) \) the number of integers \( n \) such that
For any $\epsilon > 0$, $A(x) < 2.2x^{1/2} 2^{(1+\epsilon) \log x / \log \log x}$ for "large" $x$.

Proof. We use an argument of Kanold [5]. A powerful number is a positive integer $m$ with the property that if $p|m$ then $p^2|m$. It is obvious that every positive integer can be written uniquely in the form $N_p N_F$ where $(N_p, N_F) = 1$, $N_p$ is powerful (or 1) and $N_F$ is square-free. If $P(x)$ is the number of powerful numbers not exceeding $x$ it is proved in [2] that $P(x) \sim cx^{3/2}$ where $c = \zeta(3/2)/\zeta(3) = 2.173 \ldots$. It follows that $P(x) < 2.2x^{1/2}$ for large $x$.

If $N_p$ is a (fixed) powerful number let $g(N_p, x)$ denote the number of square-free numbers $N_F$ such that $(N_p, N_F) = 1$, $N_p N_F \leq x$, and $N_p N_F \in UH$. If $G(x) = \max \{g(N_p, x)\}$ for $N_p \leq x$ it follows that

$$A(x) < 2.2x^{1/2} G(x) \quad \text{for large } x.$$  

We now investigate the magnitude of $G(x)$. Let $N_p$ be a powerful number for which square-free numbers $m_1, m_2, \ldots, m_{G(x)}$ exist such that $(N_p, m_i) = 1$, $N_p m_i \leq x$ and $N_p m_i \in UH$ for $i = 1, 2, \ldots, G(x)$. Then

$$H^*(N_p m_i) = H^*(N_p) H^*(m_i) = Z_i,$$

where $Z_i$ is an integer for $i = 1, \ldots, G(x)$. If $Z_i = Z_j$ where $i \neq j$, and $(m_i, m_j) = d$ then, of course, $H^*(M_i) = H^*(M_j)$ where $M_i = m_i/d$ and $M_j = m_j/d$. If $M_i = p_1 \cdots p_s$ (or 1) and $M_j = q_1 \cdots q_t$ (where $p_1 < \cdots < p_s$, $q_1 < \cdots < q_t$, and $p_u \neq q_v$) then from (3):

$$2^s p_1 \cdots p_s (1 + q_1) \cdots (1 + q_t) = 2^t q_1 \cdots q_t (1 + p_1) \cdots (1 + p_s).$$

It is not difficult to see that $p_s \leq 3$ and $q_t \leq 3$ so that (assuming $M_i < M_j$)

$$M_j = 6, \quad M_i = 1; \quad M_j = 3, \quad M_i = 1 \quad \text{or} \quad 2; \quad M_j = 2, \quad M_i = 1$$

are the only logical possibilities. Since none of these satisfies (5) we conclude that $Z_i \neq Z_j$ unless $i = j$. Therefore, without loss of generality, $Z_1 < Z_2 < \cdots < Z_{G(x)}$ so that $G(x) \leq Z_{G(x)} = H^*(N_p m_{G(x)}) < 2^k$, where $k$ is the number of prime factors in $N_p m_{G(x)}$. If $N = 2 \cdot 3 \cdot 5 \cdots p_K$ is the "longest prime product" not exceeding $x$ then $k \leq K$. Since $K \sim \log N/\log \log N$ (see [4, §22.10]) it follows that if $\epsilon > 0$ then

$$G(x) < 2^{(1+\epsilon) \log x / \log \log x} \quad \text{for large } x.$$  

Our theorem follows from (4) and (6).

Remark. It follows easily from Theorem 3 that $UH$ has zero density.

That the set of unitary perfect numbers has density zero was first shown by Subbarao [3, p. 1117].
Table 1. The unitary harmonic numbers in \([1, 10^{6}]\).

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