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ON THE CAUCHY PROBLEM OF THE DIFFERENTIAL OPERATOR S_μ

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ABSTRACT. I. M. Gelfand and G. E. Shilov have obtained the uniqueness and correctness class of the Cauchy problem of the differential operator $i(\partial/\partial x)$. If S_μ is some particular differential operator, then the uniqueness class of the differential operator S_μ is given in this paper.

According to [10, pp. 679–682], we define the spaces $B_{\mu,b}$ and $\mathcal{Y}^{2q}_{\mu,b}$ as follows:

Definition 1. $\phi \in B_{\mu,b}$ if ϕ is a smooth function, $\phi(x) = 0$ for $x > b$, and

$$\gamma^\mu_{b,k}(\phi) \triangleq \sup_{0<x<\infty} |(x^{-1}D)^k(x^{-\mu-1/2}\phi(x))| < \infty, \qquad k = 0, 1, 2, \cdots,$$

where $b > 0$ is a constant and μ is any real number.

Definition 2. For each $q = 1, 2, 3, \cdots$, $\Phi \in \mathcal{Y}^{2q}_{\mu,b}$ if $z^{-\mu-1/2}\Phi$ is an even entire function and

$$\alpha^{\mu,2q}_{b,k}(\Phi) \triangleq \sup_{z=x+iy} |e^{-b|y|^{2q}}(z^{2k-\mu-1/2}\Phi(z))| < \infty, \qquad k = 0, 1, 2, \cdots,$$

where $b > 0$ is a constant and μ is any real number. Here $z^{-\mu-1/2}$ is understood to be a principal value.

The topology of the spaces $B_{\mu,b}$ and $\mathcal{Y}^{2q}_{\mu,b}$ is generated by the seminorms $\{\gamma^\mu_{b,k}\}_{k=0}^\infty$ and $\{\alpha^{\mu,2q}_{b,k}\}_{k=0}^\infty$ respectively. It follows from Definitions 1 and 2 that $B_{\mu,b}$ and $\mathcal{Y}^{2q}_{\mu,b}$ are Fréchet spaces. If we define

$$\sigma^\mu_{b,k}(\phi) = \max_{0\le\nu\le k} \gamma^\mu_{b,\nu}(\phi), \qquad \beta^{\mu,2q}_{b,k}(\Phi) = \max_{0\le\nu\le k} \alpha^{\mu,2q}_{b,\nu}(\Phi)$$

then $\sigma^\mu_{b,k}$ and $\beta^{\mu,2q}_{b,k}$ define a norm on the spaces $B_{\mu,b}$ and $\mathcal{Y}^{2q}_{\mu,b}$ respectively.

For $\mu \ge -1/2$, let $\mathcal{H}_\mu$ be the Hankel transformation defined by

$$\mathcal{H}_\mu\phi(x) = \int_0^\infty \phi(x)(xy)^{1/2}J_\mu(xy)\,dx \tag{1}$$

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where $J_\mu(xy)$ is the Bessel function of the first kind; then an application of Griffith's theorem [3, pp. 109–115] and Zemanian's theorem [10, pp. 683–684] yields

Theorem 1. *For $\mu \geq -\frac{1}{2}$ and for each $q = 1, 2, 3, \cdots$, the Hankel transformation $\mathcal{H}_\mu$ is an isomorphism from the space $B_{\mu,b}$ onto the space $\mathcal{Y}^{2q}_{\mu,b}$.*

Consider the differential operator N_μ defined by [11, p. 135]

$$N_\mu = x^{\mu+\frac{1}{2}} D_x x^{-(\mu+\frac{1}{2})}.$$

Then the inverse N_μ^{-1} is given by

$$N_\mu^{-1}\phi(x) = x^{\mu+\frac{1}{2}} \int_\infty^x y^{-(\mu+\frac{1}{2})}\phi(y)\,dy.$$

Let m be any nonnegative integer such that $\mu + m \geq -\frac{1}{2}$ for any fixed real number μ. Define the Hankel transformation $\mathcal{H}_{\mu,m}$ for $\phi \in B_{\mu,b}$ by [11, p. 163]

$$\Phi(y) \triangleq \mathcal{H}_{\mu,m}\phi(x) = (-1)^m y^{-m} \mathcal{H}_{\mu+m} N_{\mu+m-1} \cdots N_{\mu+1} N_\mu \phi(x). \tag{2}$$

Then the inverse Hankel transformation $\mathcal{H}^{-1}_{\mu,m}$ is defined for $\Phi \in \mathcal{Y}^{2q}_{\mu,b}$ by

$$\phi(x) \triangleq \mathcal{H}^{-1}_{\mu,m}\Phi(y) = (-1)^m N_\mu^{-1} N_\mu^{-1} \cdots N_{\mu+m-1}^{-1} \mathcal{H}_{\mu+m} y^m \Phi(y). \tag{3}$$

Note that N_μ is an isomorphism from $B_{\mu,b}$ onto $B_{\mu+1,b}$ and N_μ^{-1} is an isomorphism from $B_{\mu+1,b}$ onto $B_{\mu,b}$. Thus Theorem 1 leads us to

Theorem 2. *For any real number μ, the Hankel transformation $\mathcal{H}_{\mu,m}$ defined by (2) is an isomorphism from the space $B_{\mu,b}$ onto the space $\mathcal{Y}^{2q}_{\mu,b}$ where m is a nonnegative integer such that $\mu + m \geq -\frac{1}{2}$. Moreover the inverse Hankel transformation $\mathcal{H}^{-1}_{\mu,m}$ is given by (3).*

Theorem 2 was first proved by Koh [14, Lemma 3, p. 324]. Let f and ϕ belong to the spaces $(\mathcal{Y}^{2q}_{\mu,b})'$ and $B_{\mu,b}$ respectively. Then the generalized Hankel transformation $\mathcal{H}'_\mu$ for any real number μ is defined by

$$\langle \mathcal{H}'_\mu f, \phi \rangle = \langle f, \mathcal{H}_{\mu,m}\phi \rangle \tag{4}$$

where m is a nonnegative integer chosen as before. From Theorem 2 and equation (4), we have

Theorem 3. *For any real number μ, the generalized Hankel transformation $\mathcal{H}'_\mu$ is an isomorphism from the dual space $(\mathcal{Y}^{2q}_{\mu,b})'$ onto the dual space $(B_{\mu,b})'$.*

The following lemma is an easy consequence of Definition 2.

Lemma 1. *For any real number $a > 0$ and for each $q = 1, 2, 3, \cdots$, let Ψ be an even entire function such that*

$$|\Psi(z)| \leq Ce^{a|y|^{2q}}(1 + |z|^{2m})$$

where C is any positive constant and m is any nonnegative integer. Then $\Phi \mapsto \Psi\Phi$ is a continuous linear mapping from the space $\mathcal{Y}^{2q}_{\mu,b}$ into $\mathcal{Y}^{2q}_{\mu,a+b}$.

Let $\mathcal{S}_e(R^2)$ be the space of even smooth functions of rapid descent. Let $\mathcal{S}_{e,\mu}(z)$ be the subspace of the functions $z^{\mu+1/2}f$ where $f \in \mathcal{S}_e(R^2)$. Then the following algebraic inclusion relations are apparent:

$$\mathcal{S}_{e,\mu}(z) \subset \mathcal{Y}^{2q}_{\mu,b} \subset \mathcal{Y}^{2q}_{\mu,a+b} \subset z^{\mu+1/2} \cdot \mathcal{S}_e(R^2).$$

Observe that $\mathcal{S}_{e,\mu}(z)$ and $\mathcal{S}_e(R^2)$ carry their Fréchet space topologies. Note also that the topology of $\mathcal{Y}^{2q}_{\mu,b}$ is identical with the induced topology by $\mathcal{Y}^{2q}_{\mu,a+b}$. Since $\mathcal{S}_{e,\mu}(z)$ is dense in $z^{\mu+1/2} \cdot \mathcal{S}_e(R^2)$ [7, Theorem 15.5, p. 160], we immediately get

Lemma 2. *For any two positive real numbers a and b, the space $\mathcal{Y}^{2q}_{\mu,b}$ is dense in $\mathcal{Y}^{2q}_{\mu,a+b}$ for each $q = 1, 2, 3, \cdots$.*

Theorems 1 and 2 and Lemma 2 yield

Lemma 3. *For any real number μ and for any two positive real numbers a and b, the space $B_{\mu,b}$ is dense in $B_{\mu,a+b}$.*

Consider now the Cauchy problem of the differential operator $S_\mu \overset{\Delta}{=} \partial^2/\partial x^2 - (4\mu^2 - 1)/4x^2$:

$$\partial u(x, t)/\partial t = P(S_\mu)u(x, t), \tag{5}$$

$$u(x, 0) = u_0(x) \tag{6}$$

where $u(x, t)$ is an $m \times 1$ column vector, and P is an $m \times m$ polynomial matrix with constant coefficients.

Theorem 4. *Let μ be a real number $\geq -\frac{1}{2}$. Then the Cauchy problem (5)–(6) admits a unique solution $u(x, t)$ in the dual space $(B_{\mu,a+b})'$ for the interval $0 \leq t \leq T$, $T = t_0 + (a + b)c_0^{-1}2^{-4p_0}$ and for any initial function*

$u_0(x) \in (B_{\mu,a+b})'$ where p_0 is the reduced order of the system (5)–(6) with S_μ replaced by $i(\partial/\partial x)$ and c_0 is a constant depending on P.

Proof. From the fundamental theorem [2, pp. 35–48], the Cauchy problem (5)–(6) has a unique solution in a dual space E' if the Cauchy problem:

$$\partial\phi(x, t)/\partial t = \tilde{P}(S_\mu)\phi(x, t), \tag{7}$$

$$\phi(x, t_0) = \phi_0(x) \tag{8}$$

has a solution in the space E where $\tilde{P}$ is the adjoint of p. In our case E turned out to be $B_{\mu,a+b}$. Let μ be a real number $\geq -\frac{1}{2}$. From [8, p. 565] or [9, p. 686], the Hankel transformation $\mathcal{H}_\mu$ of (7)–(8) gives

$$\partial\Phi(y, t)/\partial t = \tilde{P}(-y^2)\Phi(y, t), \tag{9}$$

$$\Phi(y, t_0) = \Phi_0(y) \tag{10}$$

where $\Phi(y, t) = \mathcal{H}_\mu\phi(x, t)$. Obviously the formal solution of (9)–(10) is given by

$$\Phi(y, t) = \exp[(t - t_0)\tilde{P}(-y^2)]\Phi_0(y).$$

Define $Q(\eta, t_0, t) = \exp[(t - t_0)\tilde{P}(-\eta^2)]$ where $\eta = u + iv$. Let p_0 be the reduced order of the system (5)–(6) where S_μ is replaced by $i(\partial/\partial x)$. Upon utilizing [2, p. 53], we obtain

$$\|Q(\eta, t_0, t)\| \leq C(1 + |\eta|^2)^{p_0(m-1)}\exp[c_0|t - t_0|\,|\eta|^{4p_0}]$$

where c_0 is a constant depending on p_0. Since $|\eta|^{4p_0} \leq 2^{4p_0}(|u|^{4p_0} + |v|^{4p_0})$, it follows that

$$\|Q(\eta, t_0, t)\| \leq C_1\exp[c_0 2^{4p_0}|t - t_0|\,|u|^{4p_0}]\exp[c_0 2^{4p_0}|t - t_0|\,|v|^{4p_0}]. \tag{11}$$

Choose a positive integer q greater than $2p_0$. Applying the Phragmén-Lindelöf theorem [6, pp. 176–181] in the domain $-\pi/2 + \epsilon \leq \arg\eta \leq \pi/2 - \epsilon$, $\epsilon > 0$, and Lemma 1, we can see from (11) that $\Phi \mapsto Q(\eta, t_0, t)$ is a continuous linear mapping from the space $\mathcal{Y}^{2q}_{\mu,b}$ into $\mathcal{Y}^{2q}_{\mu,a+b}$ where t is chosen such that $|t - t_0| \leq c_0^{-1}2^{-4p_0}(a + b)$. Thus the Cauchy problem (9)–(10) has a unique solution in $\mathcal{Y}^{2q}_{\mu,a+b}$. Since $\mathcal{H}_\mu^{-1}\mathcal{Y}^{2q}_{\mu,a+b} = B_{\mu,a+b}$ for $\mu \geq -\frac{1}{2}$ from Theorem 1, and since $B_{\mu,a}$ is dense in $B_{\mu,a+b}$ from Lemma 3, the Cauchy problem (7)–(8) has a unique solution in $B_{\mu,a+b}$. It follows from the fundamental theorem [2, pp. 35–48] that the Cauchy problem (5)–(6) admits a unique solution in the dual space $(B_{\mu,a+b})'$ for the interval $0 \leq t \leq T$, $T = c_0^{-1} 2^{-4p_0}(a+b)$. This completes the proof.

For $\mu < -\frac{1}{2}$, an application of $\mathcal{H}_{\mu,m}$ defined by (2) instead of $\mathcal{H}_\mu$ in the above proof shows us

Corollary 1. *Theorem 4 holds for whatever the choice of μ may be.*

Remark 1. If p_0 is the reduced order of the system (5)–(6) with S_μ replaced by $i(\partial/\partial x)$, we have to select q greater than $2p_0$ so that $\Phi \mapsto Q(\eta, t_0, t)\Phi$ is a continuous linear mapping from $\mathcal{Y}^{2q}_{\mu,b}$ into $\mathcal{Y}^{2q}_{\mu,a+b}$.

Remark 2. Let p_0 be defined as in Remark 1. Then the reduced order of the system (5)–(6) is $2p_0$.

Remark 3. If the coefficients of the system (5)–(6) depend on t, upon utilizing [2, pp. 58–60], it is not hard to verify the validity of Theorem 4 and Corollary 1 in this case too.

Problem. Let h be the genus of the system (5)–(6) with respect to the differential operator $i(\partial/\partial x)$ [2, pp. 107–115]. Then what is the genus of the same system with $i(\partial/\partial x)$ replaced by S_μ? We conjecture the answer to be $2h$. Once this problem is solved, we may be able to give a correctness class of the system (5)–(6).

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