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STRONG REGULARITY AT NONPEAK POINTS

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ABSTRACT. We construct a uniform algebra which is strongly regular at a nonpeak point.

1. Let A be a uniform algebra and x be a point in $M(A)$, the maximal ideal space of A. We denote by M_x the maximal ideal at x and by N_x the ideal of functions vanishing in a neighborhood of x. We say that A is *strongly regular at* x if N_x is uniformly dense in M_x. It has been conjectured that if A is strongly regular at x, then x is a peak point. In a recent paper, Chalice [2] constructed a nontrivial uniform algebra which is strongly regular at every point in a dense subset E of $M(A)$ while each point of E is a peak point. In this note, we construct a uniform algebra which is strongly regular at a nonpeak point, and thus give a negative answer to this conjecture.

2. Let X be a compact subset of the plane. We denote by $R(X)$ the uniform closure of all the (restrictions to X of) rational functions having no poles on X. We note that $R(X)$ is strongly regular at a point x in X if and only if the function $z - x$ belongs to $\overline{N}_x$, the uniform closure of N_x, since the ideal $\{(z-x)f: f \in R(X)\}$ is uniformly dense in M_x.

Throughout this note, $\Delta(a, r)$ will denote the open disk with center a and radius r; E' will denote the complement of a set E.

We will need the following lemma, due to McKissick [3]:

Lemma. *Let D be the open unit disk. There is a sequence $\{a_k\}$ in D, $0 < |a_k| \leq |a_{k+1}| \to 1$, such that, for any $\epsilon > 0$, there is a sequence $\{\Delta(a_k, r_k)\}$ in D and a sequence $\{f_n\}$ of rational functions with the following properties*:

(1) $\sum_1^\infty r_k < \epsilon$.

(2) *The poles of f_n lie in $\{a_1, \ldots, a_n\}$.*

(3) *$f_n \to F$ uniformly on $(\bigcup_k \Delta(a_k, r_k))'$ and $F \equiv 0$ on D' while $F(0) = 1$.*

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Theorem. *There is a uniform algebra which is strongly regular at a nonpeak point.*

Proof. We choose a sequence $\{\epsilon_m\}$ of positive numbers with $\sum \epsilon_m < |a_1|$. For each fixed m, let $\{\Delta(a_k, r_{m,k})\}_{k=1}^{\infty}$ and $\{f_{m,n}\}_{n=1}^{\infty}$ be chosen as in the Lemma so that

(1) $\sum_{k=1}^{\infty} r_{m,k} < \epsilon_m$.

(2) The poles of $f_{m,n}$ lie in $\{a_1, \ldots, a_n\}$.

(3) $f_{m,n} \longrightarrow F_m$ uniformly on $(\bigcup_k \Delta(a_k, r_{m,k}))'$ as $n \longrightarrow \infty$ and $F_m \equiv 0$ on D' while $F_m(0) = 1$.

We also choose $\delta_m > 0$ such that $|\omega| < \delta_m$ implies $|F_m(\omega) - 1| < \epsilon_m$. Let $\{D_k^m\}_{k=1}^{\infty}$ be the images of $\{\Delta(a_k, r_{m,k})\}_{k=1}^{\infty}$ under the linear fractional transformation $L_m: \omega \longrightarrow z = \delta_m \epsilon_m \|F_m\|^{-1} \omega^{-1}$. We set $X = \overline{D} \setminus \bigcup_{m,k=1}^{\infty} \{D_k^m\}$. It is easy to compute that

$$\sum_{m,k=1}^{\infty} (\text{radius of } D_k^m)/|\text{center of } D_k^m| = \sum_{m,k=1}^{\infty} \frac{r_{m,k}}{|a_k|} < \left(\sum \epsilon_m\right) \Big/ |a_1| < 1.$$

Thus 0 is a nonpeak point for $R(X)$ since there exists a representing measure μ for 0 with $\mu(\{0\}) = 0$ (see, e.g., [1, p. 99]).

We now prove that $R(X)$ is strongly regular at 0. If we set $g_m(z) = zF_m(L_m^{-1}(z))$ and $U_m = \{z: |z| < \epsilon_m/\|F_m\|\}$, then $g_m \in N_0$. We have $|g_m - z| = |F_m(L_m^{-1}(z)) - 1| \cdot |z|$. This is no more than ϵ_m on $U_m' \cap X$, and it is no more than $(\|F_m\| + 1)(\epsilon_m/\|F_m\|)$ on U_m, which implies that $z \in \overline{N}_0$ (we may assume $\{\|F_m\|\}$ is nondecreasing), so the conclusion follows from an early remark.

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