COMPACT OPERATORS IN THE ALGEBRA GENERATED BY ESSENTIALLY UNITARY $C_0$ OPERATORS

ERIC A. NORDGREN

ABSTRACT. It will be shown that the compact operators in the weakly closed algebra generated by an essentially unitary $C_0$ contraction are weakly dense in the algebra. The result implies the extension of a double dual theorem of Kriete, Moore and Page and yields a partial answer to a question on reductive algebras raised by Rosenthal.

A contraction $T$ on a separable Hilbert space is called essentially unitary in case both $1 - T^*T$ and $1 - TT^*$ are compact. A technique, based on Muhly's characterization [6] of compact operators commuting with a $C_{00}$ operator, was introduced in [5] for showing that the weakly closed algebra $\mathcal{A}_T$ generated by an essentially unitary $C_0$ contraction $T$ and the identity contains nonzero compact operators. That technique will be used here to show that the identity, and hence every operator in $\mathcal{A}_T$, is in fact the weak limit of a sequence of compact operators in $\mathcal{A}_T$. Two consequences of this result are to be derived.

Kriete, Moore and Page [4] showed that if $T$ is the compression of a simple shift operator to the orthogonal complement of one of its nontrivial invariant subspaces, then the commutant $\mathcal{A}_T'$ of $T$ may be identified with the second dual of the Banach space of compact operators in $\mathcal{A}_T'$. (Actually their theorem is more general as it deals with intertwining operators between pairs of such compressions.) Since a compression of the simple shift to the orthogonal complement of one of its nontrivial invariant subspaces is a $C_0$ operator (see [9]) whose defect operators are rank one, we see that the result of Kriete, Moore and Page is a special case of Corollary 1 below in which $T$ is merely required to be an essentially unitary $C_0$ contraction. Thus, as they conjectured, considerable generalization of their result is possible.
Rosenthal studies several questions related to the reductive algebra problem (see [7]) in [8], and among other things obtains the result that if a compact operator commutes with a reductive algebra, then its adjoint does also. He raises the question of which operators besides compact ones have this property and asks in particular if parts of finite-multiplicity backwards shifts have this property. For $C_0$ operators Corollary 3 goes beyond this somewhat and shows that essentially unitary $C_0$ operators have this property.

**Theorem.** If $T$ is an essentially unitary $C_0$ operator, then the weakly closed algebra generated by $T$ and the identity contains a sequence of compact operators that converges weakly to the identity.

**Proof.** Let $T$ be an essentially unitary $C_0$ contraction, and let $\mathcal{U}_T$ be the smallest weakly closed algebra that contains $T$ and the identity. As usual, $m_T$ is the minimal function of $T$, i.e. the greatest common inner divisor of the set of functions $\phi$ in $H^\infty$ such that $\phi(T) = 0$. It is shown in [5] (see the proof of Theorem 1) that if $\gamma$ is an inner divisor of $m_T$ whose closed support $K$ on the unit circle $T$ has Lebesgue measure zero, if $\psi$ is an outer function that is continuous (on $T$) and vanishes on $K$, and if $\phi = \psi m_T/\gamma$, then $\phi(T)$ is compact. We will produce a bounded sequence of such functions $\phi_n$ that converges to 1 at each point of the open unit disc $D$. Then each $\phi_n(T)$ is compact by the above, and $\{\phi_n(T)\}$ converges weakly to the identity operator (see [9, p. 114, Theorem III. 2.1. c]).

Let $\{a_k\}$ be the sequence of zeros, repeated according to multiplicity, of $m_T$, and let $\nu$ be the singular measure that determines the singular factor of $m_T$ (see [3, Chapter 5]). If $F_0$ is a subset of $T$ of Lebesgue measure zero such that $\nu(F_0) = \nu(T)$, then by regularity of $\nu$, there exists a sequence of compact subsets $K_n$ of $F_0$ such that $\lim_{n \to \infty} \nu(K_n) = \nu(F_0)$. For each $n$ define a measure $\nu_n$ on $T$ by $\nu_n(F) = \nu(F \cap K_n)$, and let $\gamma_n$ be the inner function whose zeros are the first $n$ terms of $\{a_k\}$ and whose singular factor is determined by $\nu_n$. Thus $\gamma_n$ is continuous (even analytic) on $T \setminus K_n$ [3, p. 68]. It follows that $\{\gamma_n\}$ converges to $m_T$ on $D$, and hence $\{m_T \setminus \gamma_n\}$ converges to 1.

To obtain $\psi_n$ we will recall a portion of Fatou’s construction of a bounded analytic function on the disc that is continuous on the closed disc and vanishes on a closed subset of $T$ having Lebesgue measure zero. In [1, pp. 343–345] Fatou first constructs a nonnegative function $\phi$ on an interval such that $\phi$ is continuously differentiable on the complement of a closed set $E$ of Lebesgue measure zero and tends to infinity at each point
of $E$. The function $\phi$ is also integrable, and it is easy to see that the first few terms of the sequence $\{\phi_n\}$ from which $\phi$ is constructed can be modified in order to make $\int \phi(x) \, dx$ smaller than any preassigned positive $\epsilon$. Taking the Poisson integral $h$ of such a function $\phi$ on $[0, 2\pi]$, Fatou then obtains a nonnegative harmonic function on $D$ that has $\phi$ as its radial limit function. Further, by the differentiability of $\phi$ on the complement of $E$, the harmonic conjugate $k$ of $h$ satisfying $k(0) = 0$ has continuous radial limits at all points $e^{it}$ with $t$ not in $E$. Thus if $\psi = \exp(-b + ik)$, then $\psi$ is analytic on $D$, continuous on $\overline{D}$, and $\log|\psi(e^{it})| = -\phi(t)$ for each $t$. It follows that $\psi(e^{it}) = 0$ if $t \in E$.

The above construction may be used to obtain $\psi_n$ vanishing on $K_n$ and such that $\int \log|\psi_n| \, dm > -1/n$, where $m$ is normalized Lebesgue measure on $T$. (Here is where the fact that $\int_0^{2\pi} \phi(t) \, dt$ can be made arbitrarily small is used.) If

$$H_z(e^{it}) = \frac{(e^{it} + z)(e^{it} - z)}{e^{it} - z},$$

then $H_z$ is continuous on $T$ for each fixed $z$ in $D$, and it follows that

$$\lim_{n \to \infty} \int H_z \log|\psi_n| \, dm = 0.$$

Consequently $\lim_{n \to \infty} \psi_n(z) = 1$, since $\psi_n(z) = \exp(\int H_z \log|\psi_n| \, dm)$, and the construction of $\{\psi_n\}$ is complete.

Taking $\phi_n = \psi_n m_T / \gamma_n$, we obtain the required sequence, and the Theorem is proved.

Let $T$ be an essentially unitary $C_0$ contraction, let $\mathcal{A}_T$ be the smallest weakly closed algebra of operators containing $T$ and the identity, let $\mathcal{A}_T'$ be the commutant of $\mathcal{A}_T$, and let $\mathcal{K}_T$ be the set of compact operators in $\mathcal{A}_T$. A result of Gellar and Page [2] states that if there exists a sequence of compact operators in the commutant of an operator $A$ that converges weakly to the identity operator, then $\mathcal{A}_A'$ may be identified in a natural way with the second dual of the Banach space of compact operators in $\mathcal{A}_A'$. Hence the Theorem allows us to apply this result to obtain the following generalization of the result of Kriete, Moore and Page:

**Corollary 1.** The algebra $\mathcal{A}_T'$ may be identified with the second dual of the Banach space of compact operators in $\mathcal{A}_T'$.

**Corollary 2.** Every operator in $\mathcal{A}_T$ is a weak limit of a sequence of compact operators in $\mathcal{A}_T'$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. This follows immediately from the Theorem.

Corollary 3. If $T$ is an essentially unitary $C_0$ contraction in the commutant of a reductive algebra $\mathfrak{A}$, then $T^*$ is also in the commutant of $\mathfrak{A}$.

Proof. This follows from Corollary 2 since Rosenthal has shown [8] that every compact operator has the property asserted for $T$. For if $T \in \mathfrak{A}'$, then $\mathfrak{A}_T \subset \mathfrak{A}'$, and by Rosenthal’s result $\mathfrak{K}_T^* (= \{ K^*: K \in \mathfrak{K}_T \}) \subset \mathfrak{A}'$. It follows that $\mathfrak{A}_T^* \subset \mathfrak{A}'$, and in particular $T^* \in \mathfrak{A}'$.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NEW HAMPSHIRE 03824