PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 1, August 1975

COMPACT OPERATORS IN THE ALGEBRA GENERATED BY ESSENTIALLY UNITARY C_0 OPERATORS[1]

ERIC A. NORDGREN

ABSTRACT. It will be shown that the compact operators in the weakly closed algebra generated by an essentially unitary C_0 contraction are weakly dense in the algebra. The result implies the extension of a double dual theorem of Kriete, Moore and Page and yields a partial answer to a question on reductive algebras raised by Rosenthal.

A contraction T on a separable Hilbert space is called essentially unitary in case both $1 - T^*T$ and $1 - TT^*$ are compact. A technique, based on Muhly's characterization [6] of compact operators commuting with a C_{00} operator, was introduced in [5] for showing that the weakly closed algebra $\mathfrak{A}_T$ generated by an essentially unitary C_0 contraction T and the identity contains nonzero compact operators. That technique will be used here to show that the identity, and hence every operator in $\mathfrak{A}_T$, is in fact the weak limit of a sequence of compact operators in $\mathfrak{A}_T$. Two consequences of this result are to be derived.

Kriete, Moore and Page [4] showed that if T is the compression of a simple shift operator to the orthogonal complement of one of its nontrivial invariant subspaces, then the commutant $\mathfrak{A}_T'$ of T may be identified with the second dual of the Banach space of compact operators in $\mathfrak{A}_T'$. (Actually their theorem is more general as it deals with intertwining operators between pairs of such compressions.) Since a compression of the simple shift to the orthogonal complement of one of its nontrivial invariant subspaces is a C_0 operator (see [9]) whose defect operators are rank one, we see that the result of Kriete, Moore and Page is a special case of Corollary 1 below in which T is merely required to be an essentially unitary C_0 contraction. Thus, as they conjectured, considerable generalization of their result is possible.

Received by the editors May 23, 1974.
AMS (MOS) subject classifications (1970). Primary 47C05.
Key words and phrases. Hilbert space operator, C_0 contraction, compact defect operator, reductive algebra.
[1] Research supported in part by a grant from the National Science Foundation.

Rosenthal studies several questions related to the reductive algebra problem (see [7]) in [8], and among other things obtains the result that if a compact operator commutes with a reductive algebra, then its adjoint does also. He raises the question of which operators besides compact ones have this property and asks in particular if parts of finite-multiplicity backwards shifts have this property. For C_0 operators Corollary 3 goes beyond this somewhat and shows that essentially unitary C_0 operators have this property.

Theorem. *If T is an essentially unitary C_0 operator, then the weakly closed algebra generated by T and the identity contains a sequence of compact operators that converges weakly to the identity.*

Proof. Let T be an essentially unitary C_0 contraction, and let $\mathfrak{A}_T$ be the smallest weakly closed algebra that contains T and the identity. As usual, m_T is the minimal function of T, i.e. the greatest common inner divisor of the set of functions ϕ in H^∞ such that $\phi(T) = 0$. It is shown in [5] (see the proof of Theorem 1) that if γ is an inner divisor of m_T whose closed support K on the unit circle $\mathbf{T}$ has Lebesgue measure zero, if ψ is an outer function that is continuous (on $\mathbf{T}$) and vanishes on K, and if $\phi = \psi m_T/\gamma$, then $\phi(T)$ is compact. We will produce a bounded sequence of such functions ϕ_n that converges to 1 at each point of the open unit disc D. Then each $\phi_n(T)$ is compact by the above, and $\{\phi_n(T)\}$ converges weakly to the identity operator (see [9, p. 114, Theorem III. 2.1. c″]).

Let $\{\alpha_k\}$ be the sequence of zeros, repeated according to multiplicity, of m_T, and let ν be the singular measure that determines the singular factor of m_T (see [3, Chapter 5]). If F_0 is a subset of $\mathbf{T}$ of Lebesgue measure zero such that $\nu(F_0) = \nu(\mathbf{T})$, then by regularity of ν, there exists a sequence of compact subsets K_n of F_0 such that $\lim_{n\to\infty}\nu(K_n) = \nu(F_0)$. For each n define a measure ν_n on $\mathbf{T}$ by $\nu_n(F) = \nu(F \cap K_n)$, and let γ_n be the inner function whose zeros are the first n terms of $\{\alpha_k\}$ and whose singular factor is determined by ν_n. Thus γ_n is continuous (even analytic) on $\mathbf{T}\backslash K_n$ [3, p. 68]. It follows that $\{\gamma_n\}$ converges to m_T on D, and hence $\{m_T\backslash\gamma_n\}$ converges to 1.

To obtain ψ_n we will recall a portion of Fatou's construction of a bounded analytic function on the disc that is continuous on the closed disc and vanishes on a closed subset of $\mathbf{T}$ having Lebesgue measure zero. In [1, pp. 343–345] Fatou first constructs a nonnegative function ϕ on an interval such that ϕ is continuously differentiable on the complement of a closed set E of Lebesgue measure zero and tends to infinity at each point of E. The function ϕ is also integrable, and it is easy to see that the first few terms of the sequence $\{\phi_n\}$ from which ϕ is constructed can be modified in order to make $\int\phi(x)\,dx$ smaller than any preassigned positive ϵ. Taking the Poisson integral h of such a function ϕ on $[0, 2\pi]$, Fatou then obtains a nonnegative harmonic function on D that has ϕ as its radial limit function. Further, by the differentiability of ϕ on the complement of E, the harmonic conjugate k of h satisfying $k(0) = 0$ has continuous radial limits at all points e^{it} with t not in E. Thus if $\psi = \exp -(h + ik)$, then ψ is analytic on D, continuous on $\overline{D}$, and $\log|\psi(e^{it})| = -\phi(t)$ for each t. It follows that $\psi(e^{it}) = 0$ if $t \in E$.

The above construction may be used to obtain ψ_n vanishing on K_n and such that $\int\log|\psi_n|\,dm > -1/n$, where m is normalized Lebesgue measure on $\mathbf{T}$. (Here is where the fact that $\int_0^{2\pi}\phi(t)\,dt$ can be made arbitrarily small is used.) If

$$H_z(e^{it}) = (e^{it} + z)/(e^{it} - z),$$

then H_z is continuous on $\mathbf{T}$ for each fixed z in D, and it follows that

$$\lim_{n\to\infty} \int H_z \log|\psi_n|\,dm = 0.$$

Consequently $\lim_{n\to\infty} \psi_n(z) = 1$, since $\psi_n(z) = \exp(\int H_z \log|\psi_n|\,dm)$, and the construction of $\{\psi_n\}$ is complete.

Taking $\phi_n = \psi_n m_T/\gamma_n$, we obtain the required sequence, and the Theorem is proved.

Let T be an essentially unitary C_0 contraction, let $\mathfrak{A}_T$ be the smallest weakly closed algebra of operators containing T and the identity, let $\mathfrak{A}_T'$ be the commutant of $\mathfrak{A}_T$, and let $\mathfrak{K}_T$ be the set of compact operators in $\mathfrak{A}_T$. A result of Gellar and Page [2] states that if there exists a sequence of compact operators in the commutant of an operator A that converges weakly to the identity operator, then $\mathfrak{A}_A'$ may be identified in a natural way with the second dual of the Banach space of compact operators in $\mathfrak{A}_A'$. Hence the Theorem allows us to apply this result to obtain the following generalization of the result of Kriete, Moore and Page:

Corollary 1. *The algebra $\mathfrak{A}_T'$ may be identified with the second dual of the Banach space of compact operators in $\mathfrak{A}_T'$.*

Corollary 2. *Every operator in $\mathfrak{A}_T$ is a weak limit of a sequence of compact operators in $\mathfrak{A}_T$.*

Proof. This follows immediately from the Theorem.

Corollary 3. *If T is an essentially unitary C_0 contraction in the commutant of a reductive algebra $\mathfrak{A}$, then T^* is also in the commutant of $\mathfrak{A}$.*

Proof. This follows from Corollary 2 since Rosenthal has shown [8] that every compact operator has the property asserted for T. For if $T \in \mathfrak{A}'$, then $\mathfrak{A}_T \subset \mathfrak{A}'$, and by Rosenthal's result $\mathfrak{K}_T^* (= \{K^*: K \in \mathfrak{K}_T\}) \subset \mathfrak{A}'$. It follows that $\mathfrak{A}_T^* \subset \mathfrak{A}'$, and in particular $T^* \in \mathfrak{A}'$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NEW HAMPSHIRE 03824