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NONNORMAL PRODUCTS OF ω_μ-METRIZABLE SPACES

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ABSTRACT. We describe a metric space X and an ω_1-metrizable space Y whose product $X \times Y$ is not normal. This provides another example to show that the product of a metric space with a normal space need not be normal. The first example of this nature was given by E. Michael, and we compare our example to his. We use our example to answer several questions concerning the normality of finite products of linearly stratifiable spaces, and countable products of hereditarily paracompact spaces. We also use some related examples to answer a question of K. Morita concerning $P(\mathfrak{m})$-spaces.

1. **Introduction and statement of results.** Let ω and ω_1 denote the first infinite and the first uncountable ordinals respectively. Let $D_1 = \omega_1$ with the discrete topology. Since D_1 is a discrete space, the countable product D_1^ω with the usual product topology is a metric space. Let D_1^* denote the set $\omega_1 + 1$ with the smallest topology larger than the order topology such that all ordinals $\alpha < \omega_1$ are isolated, and let $\mathbf{B}_1$ denote the product set $(D_1^*)^\omega$ with the box topology. (The collection of all subsets of $(D_1^*)^\omega$ of the form $\bigcap_{i=0}^\infty \pi_i^{-1}(U_i)$, where U_i is open in D_1^* for all i, and π_i is the ith projection map, forms a base for the box topology.)

Theorem 1. *The product* $D_1^\omega \times \mathbf{B}_1$ *is not normal.*

P. Nyikos [10], [11] asked if the product of a metric space with an ω_μ-metrizable space is normal. (A space X is ω_μ-metrizable, where ω_μ is a regular infinite cardinal, if and only if there is a compatible (separated) uniformity on X which is totally ordered by reverse inclusion, and has cofinality ω_μ (see [12]).) The space D_1^* is easily seen to be ω_1-metrizable, and Nyikos has proved [11] that a countable product with the box topology of ω_μ-metrizable spaces ($\omega_\mu \geq \omega_1$) is ω_μ-metrizable. Thus, $\mathbf{B}_1$ is ω_1-metrizable. It follows from Theorem 1 that the product of a metric space with an

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ω_1-metrizable space need not be normal, and this answers Nyikos' question.

We conjectured in [16] that the product of two linearly stratifiable spaces is paracompact. Theorem 1 can be used to show that this conjecture does not hold. It is easy to see that every ω_μ-metrizable space is ω_μ-stratifiable, and, therefore, $\mathbf{B}_1$ is ω_1-stratifiable (for another proof of this we can call on Example 7.1 and Theorem 5.2D of [16]). The space D_1^ω is stratifiable because every metric space is stratifiable. Hence, the product of a stratifiable space and an ω_1-stratifiable space need not be normal. This shows that [16, Conjecture 1] does not hold. E. K. Van Douwen [1] has proved that [16, Conjecture 2] does not hold either.

H. Tamano announced [13], [14] the existence of a class of spaces which he called perfectly paracompact. We pointed out in [16] that Tamano's definition of perfectly paracompact spaces was ambiguous, and we suggested that the class of spaces that he wanted was the class of linearly stratifiable spaces. Theorem 1 in this paper, and Van Douwen's result [1] that the countable box product of metric spaces need not be normal, show that the class of linearly stratifiable spaces does not satisfy either of the product theorems which Tamano claimed for his perfectly paracompact spaces in [13] and [14].

Theorem 2. *For every finite positive integer n, $\mathbf{B}_1^n$ is homeomorphic to $\mathbf{B}_1$, and $\mathbf{B}_1^\omega$ (with the product topology) is not normal.*

E. Michael [7, Example 1] gave an example of a space Y such that for every finite n, Y^n is paracompact and Y^ω is not normal. He asked if a space Y could be found so that Y^n is hereditarily paracompact for all finite n, and Y^ω is not normal. Since $\mathbf{B}_1$ is ω_1-stratifiable, it is hereditarily paracompact (and monotonically normal [3]) by [16, Theorem 4.1]. (It was first proved by I. Juhász [4, p. 134] that ω_μ-metrizable spaces are (hereditarily) paracompact.) By Theorem 2 in this paper, every finite power $\mathbf{B}_1^n$ also has these properties and $\mathbf{B}_1^\omega$ is not normal. This gives an affirmative answer to Michael's question.

In order to facilitate discussion of the next result, we recall several definitions. Let $|A|$ denote the cardinal number of a set A. A space Y is called $[\mathfrak{m}, \mathfrak{m}]$-*compact*, where $\mathfrak{m}$ is an infinite cardinal number, provided that if $\mathfrak{U}$ is an open cover of Y with $|\mathfrak{U}| = \mathfrak{m}$, there is a subcollection $\mathfrak{U}' \subset \mathfrak{U}$ such that $\mathfrak{U}'$ covers Y and $|\mathfrak{U}'| < \mathfrak{m}$. A space Y is said to have characteristic $\mathfrak{m}$, where $\mathfrak{m}$ is a regular infinite cardinal number, if each point y in Y has a fundamental system of neighborhoods $\{N_\alpha(y) | \alpha < \mathfrak{m}\}$ such that $\alpha < \beta < \mathfrak{m}$ implies $N_\alpha(y) \supset N_\beta(y)$. Every ω_μ-metrizable space has characteristic ω_μ. In particular, when we describe the topology of $\mathbf{B}_1$ in the course of proving Theorem 1, we will show that $\mathbf{B}_1$ has characteristic ω_1.

Theorem 3. *If X is paracompact and $[\mathfrak{m}, \mathfrak{m}]$-compact, and Y is paracompact and has characteristic $\mathfrak{m}$, then $X \times Y$ is paracompact.*

The Michael line $\mathbf{M}$ and the space $\mathbf{B}_1$ have a number of common features. Both are hereditarily paracompact and monotonically normal (in fact, both are non-Archimedean spaces [8], [10] and therefore elastic spaces [15]) and both can be used to show that the product of a metric space with a "nice" normal space need not be normal. One difference between them is that, by a result of M. Katětov [5], $\mathbf{M} \times \mathbf{M}$ is not hereditarily normal, but $\mathbf{B}_1 \times \mathbf{B}_1$ is hereditarily normal, since it is ω_1-metrizable. To see another difference between $\mathbf{M}$ and $\mathbf{B}_1$, recall that Michael proved $\mathbf{M} \times \mathbf{P}$ is not normal [6], where $\mathbf{P}$ is the space of irrational numbers. This shows that the product of $\mathbf{M}$ with a separable metric space need not be normal. In contrast, the product of $\mathbf{B}_1$ with every separable metric space is normal (in fact, paracompact). This follows at once from Theorem 3, since separable metric spaces are regular, Lindelöf spaces, and ω_1-metrizable spaces are paracompact and have characteristic ω_1.

We note that Theorem 1 can be generalized. Let D_μ denote a discrete space of cardinality ω_μ, and put $B_\mu = (D_\mu)^\omega$ with the product topology. Let D_μ^* denote $\omega_\mu + 1$ with the smallest topology larger than the order topology in which all ordinals $\alpha < \omega_\mu$ are isolated, and denote by $\mathbf{B}_\mu$ the space $(D_\mu^*)^\omega$ with the box topology.

Theorem 4. *For every ordinal μ, $B_\mu \times \mathbf{B}_\mu$ is not normal.*

Proof. If ω_μ is a countable sum of smaller cardinals, then the fact that $B_\mu \times \mathbf{B}_\mu$ is not normal follows from the result of Van Douwen [1] which states that $B_0 \times \mathbf{B}_0$ is not normal. If ω_μ is not a countable sum of smaller cardinals, then the techniques used to prove Theorem 1 can be applied.

K. Morita has informed me that the above results answer a question of his concerning his class of P-spaces. For an infinite cardinal $\mathfrak{m}$, a normal space X is a $P(\mathfrak{m})$-space if and only if $X \times Y$ is normal for every metrizable space Y which has a base of cardinality $\leq \mathfrak{m}$ [9, Theorem 4.2]. Morita asked [9, p. 372] if there exist normal $P(\mathfrak{n})$-spaces which are not $P(\mathfrak{m})$-spaces for $\mathfrak{m} > \mathfrak{n} \geq \aleph_0$. The above discussion of $\mathbf{B}_1$ shows that it is a $P(\aleph_0)$-space which is not a $P(\aleph_1)$-space. To answer Morita's question in general for arbitrary $\mathfrak{m} > \mathfrak{n} \geq \aleph_0$, we use that space $\mathbf{B}_\mu$, where μ is chosen so that ω_μ is the first cardinal strictly larger than $\mathfrak{n}$. Since every metric space having a base of cardinality $\leq \mathfrak{n}$ is $[\omega_\mu, \omega_\mu]$-compact, it follows from Theorem 3 that $\mathbf{B}_\mu$ is a $P(\mathfrak{n})$-space. On the other hand, B_μ is a metric space having a base of cardinality ω_μ, and by Theorem 4, $B_\mu \times \mathbf{B}_\mu$ is not normal. Thus, $\mathbf{B}_\mu$ is not a $P(\mathfrak{m})$-space for $\mathfrak{m} \geq \omega_\mu$.

2. **Proofs.** We will use the notation established in §1. For each $x = (x_i)$ in D_1^ω, and for each positive integer n, let $n(x) = \bigcap_{i=0}^{n} \pi_i^{-1}(x_i)$. Thus, $\{n(x) | n < \omega\}$ is a fundamental system of neighborhoods of x in D_1^ω. It is convenient to consider D_1^ω as a subset of $(D_1^*)^\omega = \mathbf{B}_1$:

$$D_1^\omega = \{x \in \mathbf{B}_1 \mid x_i < \omega_1 \text{ for all } i < \omega\}.$$

Each point x in D_1^ω, when considered in the topological space $\mathbf{B}_1$, is isolated because each coordinate of x is isolated in D_1^*, and $\mathbf{B}_1$ has the box topology. For each $q \in (\mathbf{B}_1 \setminus D_1^\omega)$ and for each $\alpha < \omega_1$, let

$$\alpha(q) = [\bigcap \{\pi_i^{-1}(q_i) \mid q_i < \omega_1\}] \cap [\bigcap \{\pi_i^{-1}((\alpha, \omega_1]) \mid q_i = \omega_1\}].$$

Since q has only countably many coordinates, $\{\alpha(q) | \alpha < \omega_1\}$ is a fundamental system of neighborhoods of q such that $\alpha < \beta < \omega_1$ implies $\alpha(q) \supset \beta(q)$. This shows that $\mathbf{B}_1$ has characteristic ω_1.

Proof of Theorem 1. In order to prove that $D_1^\omega \times \mathbf{B}_1$ is not normal, we will show that there are two disjoint closed sets in the space which cannot be separated by open sets. Let

$$H = \{(x, x) \mid x \in D_1^\omega\} \quad \text{and} \quad K = D_1^\omega \times (\mathbf{B}_1 \setminus D_1^\omega).$$

This choice of H and K was suggested by the examples of Michael [6] and Van Douwen [1]. It is clear that K is closed and $H \cap K = \emptyset$. To see that H is closed, let (x, y) be a point not in H. If $y_n < \omega_1$ for all $n < \omega$, then there is a coordinate k such that $x_k \neq y_k$. Thus, $k(x) \times \{y\}$ is an open neighborhood of (x, y) missing H. If $y_n = \omega_1$ for some n, let α be a countable ordinal greater than x_n. The set $n(x) \times \alpha(y)$ is an open neighborhood of (x, y) missing H. This shows that the complement of H is open. Now let V be any open set containing K. To complete the proof, we need only show that $\overline{V} \cap H \neq \emptyset$, and so we need to find a point $x = (x_0, x_1, \ldots, x_n, \ldots)$ in D_1^ω such that $(x, x) \in \overline{V} \cap H$. To find such a point we will construct a sequence of points (p_i, q_i) in K, a sequence (m_i, α_i) in $\omega \times \omega_1$, and a sequence of ordinals $x_i < \omega_1$. Let $p_0 = (0, 0, \ldots, 0, \ldots)$ and $q_0 = (\omega_1, \omega_1, \ldots, \omega_1, \ldots)$. Since $(p_0, q_0) \in K \subset V$, there exist a positive integer m_0 and an ordinal $\alpha_0 < \omega_1$ such that the open set $m_0(p_0) \times \alpha_0(q_0) \subset V$. Let x_0 be any ordinal such that $\alpha_0 < x_0 < \omega_1$. Define $p_1 = (x_0, 0, 0, \dots, 0, \dots)$ and $q_1 = (x_0, \omega_1, \omega_1, \dots, \omega_1, \dots)$. Since $(p_1, q_1) \in K$, there exists $(m_1, \alpha_1) \in \omega \times \omega_1$ such that $m_1(p_1) \times \alpha_1(q_1) \subset V$. Pick a countable ordinal $x_1 > \max\{\alpha_1, x_0\}$. Assume we have reached the kth step in this construction, and have constructed $x_0, x_1, \dots, x_{k-1}$. Let $p_k = (x_0, x_1, \dots, x_{k-1}, 0, 0, \dots, 0, \dots)$ and $q_k = (x_0, x_1, \dots, x_{k-1}, \omega_1, \omega_1, \dots, \omega_1, \dots)$. Since $(p_k, q_k) \in K \subset V$, there exists $(m_k, \alpha_k) \in \omega \times \omega_1$ such that $m_k(p_k) \times \alpha_k(q_k) \subset V$. Pick a countable ordinal $x_k > \max\{\alpha_k, x_{k-1}\}$.

By mathematical induction, we can construct a point $x = (x_0, x_1, \dots, x_m, \dots)$ such that $(x, x) \in H$. To see that $(x, x) \in \overline{V}$, let $k < \omega$ so that $k(x) \times \{x\}$ is an arbitrary basic neighborhood of (x, x). We show that

$$k(x) \times \{x\} \cap [n_{k+1}(p_{k+1}) \times \alpha_{k+1}(q_{k+1})] \neq \emptyset.$$

It is obvious that $p_{k+1} \in k(x)$, since for $0 \le n \le k$, the nth coordinates of x and p_{k+1} are the same. Next, we note that $x \in \alpha_{k+1}(q_{k+1})$. For $0 \le n \le k$, the nth coordinates of x and q_{k+1} are the same, and for the other coordinates we have $\alpha_{k+1} < x_{k+1} < x_{k+2} < \cdots$. This shows that the point (p_{k+1}, x) is in $k(x) \times \{x\} \cap [n_{k+1}(p_{k+1}) \times \alpha_{k+1}(q_{k+1})]$, and, therefore, every neighborhood of (x, x) hits V. Thus, $(x, x) \in \overline{V} \cap H$, and this completes the proof of Theorem 1.

Proof of Theorem 2. If we consider the product topology on a countable product X^ω it is well known that $(X^\omega)^n$ is homeomorphic to X^ω for all positive integers n. Very nearly the same proof will establish this fact when X^ω has the box topology. Thus, $(\mathbf{B}_1)^n$ is homeomorphic to $\mathbf{B}_1$ for all positive integers n. We now show that $(\mathbf{B}_1)^\omega$ is not normal. Since $\mathbf{B}_1$ has a closed discrete subspace of cardinality $2^{\aleph_0}$, we may consider the discrete space D_1 as a closed subset of $\mathbf{B}_1$, and D_1^ω as a closed subset of $(\mathbf{B}_1)^\omega$. Since $(\mathbf{B}_1)^\omega$ is homeomorphic to $(\mathbf{B}_1)^\omega \times \mathbf{B}_1$, we have $D_1^\omega \times \mathbf{B}_1$ as a closed subset of $(\mathbf{B}_1)^\omega$. Since this closed subspace is not normal, $(\mathbf{B}_1)^\omega$ is not normal. This completes the proof of Theorem 2.

Proof of Theorem 3. Our proof is a slight elaboration of the standard proof of the well-known special case $m = \aleph_0$, which states that the product of a paracompact space and a compact space is paracompact. Let $\mathcal{O}$ be an open cover of $X \times Y$. For each $y \in Y$, we can find a locally finite open cover $\mathcal{G}_y$ of X such that for each G in $\mathcal{G}_y$ there is an $\alpha < m$ such that $G \times N_\alpha(y)$ is a subset of some member of $\mathcal{O}$. For each $\alpha < m$, define

$$\mathcal{G}_{y,\alpha} = \{G \in \mathcal{G}_y \mid G \times N_\alpha(y) \text{ is a subset of some member of } \mathcal{O}\},$$

and $G_{y,\alpha} = \bigcup \mathcal{G}_{y,\alpha}$. Since X is $[\mathfrak{m}, \mathfrak{m}]$-compact, there exists a set $A_y \subset \mathfrak{m}$ such that $|A_y| < \mathfrak{m}$ and $\{G_{y,\alpha} | \alpha \in A_y\}$ covers X. Since $\mathfrak{m}$ is regular, $\alpha_y = \sup A_y < \mathfrak{m}$. We set $\mathcal{H}_y = \bigcup\{\mathcal{G}_{y,\alpha} | \alpha \in A_y\}$, and note that $\mathcal{H}_y$ is a locally finite open cover of X, and that $H \times N_{\alpha_y}(y)$ is a subset of some member of $\mathcal{O}$ for all $H \in \mathcal{H}_y$. Now let $\mathcal{K} = \{K_y | y \in Y\}$ be a locally finite open refinement of $\{N_{\alpha_y}(y) | y \in Y\}$ with the property that $K_y \subset N_{\alpha_y}(y)$ for all y in Y (see [2, p. 162]). It is routine to check that $\bigcup_{y \in Y}\{H \times K_y | H \in \mathcal{H}_y\}$ is a locally finite open refinement of $\mathcal{O}$.

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