PRODUCTS OF STEINER’S QUASI-PROXIMITY SPACES

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ABSTRACT. E. F. Steiner introduced a quasi-proximity $\delta$ satisfying $A \delta B$ iff $\{x\} \delta B$ for some $x$ of $A$. The purpose of this paper is to describe the Tychonoff product of topologies in terms of Steiner’s quasi-proximities. Whenever $(X_a, \delta_a)$ is the Steiner quasi-proximity space, the product proximity on $X = \prod X_a$ can be given, by using the concept of finite coverings, as the smallest proximity on $X$ which makes each projection $\delta$-continuous.

Introduction. E. F. Steiner [2] introduced a quasi-proximity $\delta$ satisfying $A \delta B$ iff $\{a\} \delta B$ for some $a$ of $A$. This note is devoted to the study of a product proximity on $X = \prod X_a$, where each $(X_a, \delta_a)$ is the above Steiner quasi-proximity space. As F. W. Stevenson [3] pointed out, there are three equivalent definitions of a product proximity. Especially, Császár and Leader defined a product proximity by using finite coverings [3]. Unfortunately, for Steiner’s quasi-proximity, it seems difficult to us to define the product proximity in the same way as Császár and Leader. We must modify the definition of a product proximity in our case (Definition 2). We then show that the Tychonoff product topology can be induced on the cartesian product $X = \prod X_a$ in terms of the quasi-proximity mentioned above.

The reader is referred to S. A. Naimpally and B. D. Warrack [1] for definitions not given here.

Preliminary definitions and lemmas.

Definition 1. A binary relation $\delta$ defined on the power set of $X$ is called a Steiner’s or $S$-quasi-proximity on $X$ iff $\delta$ satisfies the axioms below.

(I) For every $A \subset X$, $A \delta \phi$ ($\phi$ means "not-$\delta$”).

(II) $A \delta B$ iff $\{a\} \delta B$ for some $a \in A$.

(III) $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$.

(IV) For every $x \in X$, $\{x\} \delta \{x\}$.

(V) $A \delta B$ implies that there exists a subset $C$ such that $A \delta C$ and $(X - C) \delta B$.

Received by the editors May 25, 1974.

AMS (MOS) subject classifications (1970). Primary 54E05.

Key words and phrases. Quasi-proximity, product proximity, $\delta$-continuous maps.
Remark 1. Clearly Axiom (II) is equivalent to Axiom (II') below.

(II') For an arbitrary index set \( \Lambda \),

\[
\left( \bigcup_{\lambda \in \Lambda} A_\lambda \right) \delta B \iff A_{\mu} \delta B \text{ for some } \mu \in \Lambda.
\]

Furthermore, in the \( S \)-quasi-proximity we can replace Axiom (V) with Axiom (V') below.

(V') If \( x \ll A \), then there exists a set \( B \) such that \( x \ll B \ll A \). (In general, \( P \ll Q \) means \( P \delta (X - Q) \) and \( Q \) is said to be a \( \delta \)-neighborhood of \( P \).)

In fact, it is easily seen that Axiom (V) implies Axiom (V'). Conversely we show that Axiom (V) follows from Axioms (I)--(IV) and (V'). Suppose \( A \delta B \). By Axiom (II), \( \{x\} \delta B \), i.e. \( x \ll X - B \) for each \( x \in A \). Then it follows from Axiom (V') that there is a set \( C_x \) such that \( x \ll C_x \ll X - B \) for each \( x \in A \). Since \( \{x\} \delta (X - C_x) \) for each \( x \in A \),

\[
\{x\} \delta \left( X - \bigcup_{x \in A} C_x \right)
\]

by Axiom (III).

Setting \( \bigcup_{x \in A} C_x = C \), we obtain \( A \delta (X - C) \) by Axiom (II). On the other hand, since \( C_x \delta B \) for each \( x \in A \), we have \( C \delta B \) by Axiom (II'). Thus Axiom (V) surely holds.

Let \( (X, \delta) \) be an \( S \)-quasi-proximity space. For every \( A \subset X \), we set \( c(A) = \{x \in X : \{x\} \delta A\} \). Then the operator \( c \) is a topological closure operator and so \( X \) is a topological space \([2]\). This topological space is denoted by \( (X, c) \). The proof of the following is trivial.

Lemma 1. (1) If \( A \delta B \) and \( B \subset C \), then \( A \delta C \).

(2) If \( A \delta B \) and \( A \subset C \), then \( C \delta B \).

(3) If \( A \delta B \), then \( A \cap B = \emptyset \).

Lemma 2. For subsets \( A \) and \( B \) of an \( S \)-quasi-proximity space \( (X, c) \),

\[
A \delta B \iff A \cap c(B) \neq \emptyset \iff A \delta c(B).
\]

Proof. This follows readily from Axiom (II).

The following is a direct result of Lemma 2.

Lemma 3. Every topological space \( (X, \tau) \) with the topology \( \tau \) has a
compatible S-quasi-proximity \( \delta \) defined by

\[ A \delta B \iff A \cap \overline{B} \neq \emptyset, \]

where \( \overline{B} \) denotes the \( r \)-closure of \( B \).

The following lemma shows that in S-quasi-proximity spaces a \( \delta \)-continuous mapping and a continuous mapping are equivalent.

**Lemma 4.** Let \( f \) be a mapping of an S-quasi-proximity space \((X, \delta_1)\) into an S-quasi-proximity space \((Y, \delta_2)\). Then \( f \) is \( \delta \)-continuous if and only if it is a continuous mapping of the topological space \((X, r(\delta_1))\) into the topological space \((Y, r(\delta_2))\).

**Proof.** Suppose that \( f \) is \( \delta \)-continuous and that \( x \) is any point of \( c_1(A) \). Then \( \{x\} \delta_1 A \), which implies \( f(x) \delta_2 f(A) \). It follows that \( f(x) \in c_2(f(A)) \) and so \( f(c_1(A)) \subseteq c_2(f(A)) \). (\( c_1 \) and \( c_2 \) denote the closure operators in \((X, \delta_1)\) and \((Y, \delta_2)\) respectively.) Conversely let \( f \) be continuous and let \( A \delta_1 B \). Since, by Lemma 2 \( A \cap c_1(B) \neq \emptyset \), it follows that \( f(A) \cap c_2(f(B)) \neq \emptyset \). From the continuity of \( f \), we obtain that \( f(A) \subseteq c_2(f(B)) \), so that \( f \) is \( \delta \)-continuous. Q. E. D.

**Proximity products.** In the present section we attempt to obtain a direct construction of an S-quasi-proximity product space by a proximal approach. As we stated in the introduction, we modify the definition of Császár and Leader for the product proximity.

**Definition 2.** Let \( \{(X_a, \delta_a): a \in A\} \) be an arbitrary family of S-quasi-proximity spaces. Let \( X = \prod_{a \in A} X_a \) denote the cartesian product of these spaces. A binary relation \( \delta \) on the power set of \( X \) is defined as follows:

Let \( A \) and \( B \) be subsets of \( X \). Define \( A \delta B \) iff there is a point \( x_0 \in A \) such that, for any finite covering \( \{B_i: i = 1, 2, \ldots, n\} \) of \( B \), there exists a set \( B_i \) satisfying \( P_a[x_0] \delta_a P_a[B_i] \) for each \( a \in A \), where each \( P_a \) denotes the projection from \( X \) to \( X_a \).

**Remark 2.** Leader [3] defined a product proximity as follows: For \( A, B \subseteq X \), \( A \delta B \) iff for any finite coverings \( \{A_i: i = 1, 2, \ldots, m\} \) and \( \{B_j: j = 1, 2, \ldots, n\} \) of \( A \) and \( B \) respectively, there is an \( A_i \) and a \( B_j \) such that \( P_a[A_i] \delta_a P_a[B_j] \) for each \( a \in A \). But in order to prove that \( \delta \) satisfies Axiom (II), it seems difficult to use Leader's definition for the S-quasi-proximity.

**Lemma 5.** Let each \( (X_a, \delta_a) \) be an S-quasi-proximity space and let \( A \)
and B be subsets of $X = \prod X_a$. Then $A \delta B$ implies $P_a[A] \delta_a P_a[B]$ for each $a \in \Lambda$.

**Proof.** Suppose $A \delta B$. Since $\{B\}$ itself is a finite covering of $B$, there is a point $x_0$ of $A$ such that $P_a[x_0] \delta_a P_a[B]$ for each $a \in \Lambda$. Applying Axiom (II) to each $\delta_a$, we have $P_a[A] \delta_a P_a[B]$ for each $a \in \Lambda$. Q. E. D.

It follows from Lemma 5 that each projection $P_a$ is $\delta$-continuous and hence it is also continuous by Lemma 4 if $X$ becomes an S-quasi-proximity space. Now we prove the main theorem.

**Theorem 1.** The binary relation $\delta$ given by Definition 2 is an S-quasi-proximity on the cartesian product $X$. This space $(X, \delta)$ is said to be an S-quasi-proximity product space.

**Proof.** It suffices to show that $\delta$ satisfies Axioms (I)–(IV) of Definition 1 and Axiom (V)’ of Remark 1. It is easy to see that $\delta$ satisfies Axiom (I).

Axiom (II): Suppose $A \delta B$. If $x_0 \in A$ fulfills the condition in Definition 2, then clearly $x_0 \delta B$.

Conversely suppose that $\{x_0\} \delta B$ for some $x_0$ of $A$. If $\{B_i: i = 1, 2, \ldots, n\}$ is any finite covering of $B$, then there is a set $B_i$ such that $P_a[x_0] \delta_a P_a[B_i]$ for each $a \in \Lambda$. By Definition 2, this means $A \delta B$.

Axiom (III): Suppose $A \delta B$ and let $x_0 \in A$ satisfy the condition in Definition 2. If $\{D_i: i = 1, 2, \ldots, n\}$ is any finite covering of $B \cup C$, then it is a covering of $B$ as well; hence there is an $i$ such that $P_a[x_0] \delta_a P_a[D_i]$ for each $a \in \Lambda$. Thus $A \delta (B \cup C)$.

Conversely suppose $A \delta B$ and $A \delta C$. Then for any given $x \in A$, there are finite coverings $\{D_i: i = 1, 2, \ldots, n\}$ and $\{D_j: j = n + 1, \ldots, n + p\}$ of $B$ and $C$ respectively such that

$$P_a[x] \delta_a P_a[D_i] \quad \text{for } a = t_i \in \Lambda,$$

$$P_a[x] \delta_a P_a[D_j] \quad \text{for } a = s_j \in \Lambda,$$

where $i = 1, 2, \ldots, n$ and $j = n + 1, \ldots, n + p$. Since $\{D_k: k = 1, 2, \ldots, n + p\}$ is a covering of $B \cup C$, we conclude that $A \delta (B \cup C)$.

Axiom (IV): Let $x$ be a point of $X$ and let $A$ be any set such that $x \in A$. Since $P_a[x] \in P_a[A]$ for each $a \in \Lambda$, by Lemma 1(3) we have

$$P_a[x] \delta_a P_a[A] \quad \text{for each } a \in \Lambda.$$
Equivalently $P_a[x] \ll X_a - P_a[A_i]$. Since each $\delta_a$ satisfies Axiom $(V')$, there exist $G_i$ ($i = 1, 2, \ldots, n$) such that

\[ (1) \quad P_a[x] \ll G_i \ll X_a - P_a[A_i] \quad \text{for } a = t_i \in \Lambda. \]

From the first half of (1), we have

\[ (2) \quad P_a[x] \overline{\delta}_a (X_a - G_i). \]

Now we set

\[ K_i = P_a^{-1}[X_a - G_i] = X - P_a^{-1}[G_i] \]

and set $K = \bigcup_{i=1}^{n} K_i$. It follows from (2) that

\[ P_a[x] \overline{\delta}_a P_a[K_i] \quad \text{for } a = t_i \in \Lambda, \quad i = 1, 2, \ldots, n. \]

Since $\{K_i: i = 1, 2, \ldots, n\}$ is a finite covering of $K$, we obtain $\{x\} \overline{\delta} K$. This implies

\[ (3) \quad x \ll X - K. \]

Next, from the second half of (1), we have

\[ (4) \quad G_i \overline{\delta}_a P_a[A_i] \quad \text{for some } a = t_i, \quad i = 1, 2, \ldots, n. \]

On the other hand, since

\[ X - K = \bigcap_{j=1}^{n} P_a^{-1}[G_j] \quad (a = t_j), \]

it follows that

\[ P_a[X - K] = P_a \left\{ \bigcap_{j=1}^{n} P_a^{-1}[G_j] \right\} \subset G_i \quad \text{for } a = t_i. \]

Hence for every point $y$ of $X - K$,

\[ P_a[y] \in G_i \quad (a = t_i; \ i = 1, 2, \ldots, n). \]

By (4) and Lemma 1(2), we have therefore $P_a[y] \overline{\delta}_a P_a[A_i]$ for every $y$ of $X - K$, where $a = t_i; \ i = 1, 2, \ldots, n$. Because $\{A_i: i = 1, 2, \ldots, n\}$ is a finite covering of $(X - A)$, we get that

\[ (5) \quad (X - K) \overline{\delta} (X - A), \quad \text{that is, } \ X - K \ll A. \]

Relations (3) and (5) together show that $\overline{\delta}$ satisfies Axiom $(V')$. This completes the proof.
In view of Lemma 4, the following theorem shows that the Tychonoff product topology can be induced on an $S$-quasi-proximity product space $(X, \pi(\delta))$.

**Theorem 2.** The $S$-quasi-proximity $\delta$ on $X$ given by Definition 2 is the smallest $S$-quasi-proximity for which each projection $P_a$ is $\delta$-continuous.

**Proof.** Let $\beta$ be an arbitrary $S$-quasi-proximity on $X$ such that each projection $P_a$ is a $\delta$-continuous mapping of $(X, \beta)$ into $(X_a, \delta_a)$. Then we must show that $A \beta B$ implies $A \delta B$ for $A, B \subset X$. By Axiom (II), there is a point $x_0$ of $A$ such that $\{x_0\} \beta B$. Given any finite covering $\{B_i: i = 1, 2, \ldots, n\}$ of $B$, we can choose a set $B_i$ such that $\{x_0\} \beta B_i$ by Axiom (III). Since each $P_a$ is $\delta$-continuous, $P_a[x_0] \delta_a P_a[B_i]$ for each $a \in \Lambda$. Because of Definition 2, we can conclude $A \delta B$. Q. E. D.

Finally, the author would like to thank the referee who indicated the revision of Definition 2 and the proof of Theorem 1.

**REFERENCES**

