A FIXED POINT CRITERION FOR COMPACT $T_2$-SPACES

LOUIS A. TALMAN

ABSTRACT. We prove a fixed point theorem which has as conseque-
ces some theorems of W. G. Dotson, Jr., [2], [3] and of K. Baron and J.
Matkowski [1].

1. Introduction and definitions. In recent articles, W. G. Dotson, Jr.,
[2], [3] as well as K. Baron and J. Matkowski [1] have extended the Banach
contraction principle to nonexpansive self-mappings of certain classes of
compact metric spaces. The purpose of this paper is to present an extension
of the methods of these authors to certain self-mappings of arbitrary compact
$T_2$-spaces.

Let $X$ be a separated uniform space (see [5] for terminology and nota-
tion). A self-mapping of $X$ is any function from $X$ into itself. (Note that we
do not require any continuity of a self-mapping.) The symbol $X^X$ will denote
the space of all self-mappings of $X$ equipped with the topology of uniform
convergence on $X$. When $\phi \in X^X$ and $F \subseteq X^X$, we will say that $\phi$ has fixed
points modulo $F$ provided that, for each $f \in F$, at least one of the functions
$\phi \circ f$ and $f \circ \phi$ has a fixed point.

We shall use the following Proposition, which is an easy consequence
of the definitions.

Proposition. Let $\{x_\alpha : \alpha \in A\}$ be a net in a uniform space, $X$, which
converges to some point $x_0 \in X$. If $\{f_\alpha : \alpha \in A\}$ is a net in $X^X$ which con-
verges to a uniformly continuous $f$ in $X^X$, then $\{f_\alpha(x_\alpha) : \alpha \in A\}$ converges
to $f(x_0)$ in $X$.

2. The main theorem. We are now ready to give the main theorem. First,
recall that a compact $T_2$-space is, in a canonical way, a separated uniform
space, so that the notions of the previous section are meaningful in such a
space.

Theorem 1. Let $X$ be a compact $T_2$-space, and let $\phi$ be a continuous
self-mapping of $X$. If there is a net $\{f_\alpha : \alpha \in A\}$ in $X^X$ such that (i) $\phi$ has
fixed points modulo \( \{f_\alpha : \alpha \in A\} \), and (ii) \( \{f_\alpha : \alpha \in A\} \) converges to the identity function, \( \text{id}_X \), on \( X \), then \( \phi \) has a fixed point.

**Proof.** Condition (ii) requires that \( \text{id}_X \) be in the closure, in the topology on \( X^X \), of the collection \( F = \{f_\alpha : \alpha \in A\} \). If we let \( F_R = \{f_\alpha \in F : f_\alpha \circ \phi \) has a fixed point\} and \( F_L = \{f_\alpha \in F : \phi \circ f_\alpha \) has a fixed point\}, then, by (i), \( F = F_R \cup F_L \). Thus, we can select a subnet \( \{f_\beta : \beta \in B\} \) of \( F \) which converges to \( \text{id}_X \) and is entirely contained within \( F_R \) or \( F_L \). In the first case, we define \( \phi_\beta : X \to X \) for each \( \beta \in B \) by \( \phi_\beta(x) = f_\beta(\phi(x)) \); in the second case, we define \( \phi_\beta(x) = \phi(f_\beta(x)) \). In either case, for each \( \beta \in B \), there is an \( x_\beta \) in \( X \) such that \( \phi_\beta(x_\beta) = x_\beta \). Since \( X \) is compact, we may assume that \( \{x_\beta : \beta \in B\} \) converges to some \( x_0 \) in \( X \). But \( \{\phi_\beta : \beta \in B\} \) converges to \( \phi \) in \( X^X \), so application of the Proposition yields that \( x_0 = \lim x_\beta = \lim \phi_\beta(x_\beta) = \phi(x_0) \), and \( x_0 \) is a fixed point of \( \phi \).

3. Consequences. We now apply Theorem 1 in order to obtain some results due to Dotson and to Baron and Matkowski.

**Theorem 2** (Dotson [3]). Let \( S \) be a subset of a Banach space \( E \), and equip \( S \) with the relative topology from the weak topology on \( E \). Suppose that \( S \), so equipped, is compact, and that there exists a continuous function \( F : S \times [0, 1] \to S \) such that (i) \( F(s, 1) = s \) for every \( s \in S \), and (ii) there is a self-mapping, \( \phi \), of \( [0, 1] \) such that for every \( s_1, s_2 \) in \( S \) and for all \( t \) in \( [0, 1] \) we have

\[
\| F(s_1, t) - F(s_2, t) \| \leq \phi(t) \| s_1 - s_2 \|.
\]

Then any continuous function \( \psi : S \to S \) which is nonexpansive with respect to \( \| \cdot \| \) has a fixed point.

**Proof.** For each \( n = 1, 2, \ldots, \) let \( t_n = n/(n + 1) \) and define \( \phi_n : S \to S \) by \( \phi_n(s) = F(s, t_n) \). It is easy to see that \( \{\phi_n\} \) converges in \( S^S \) to \( \text{id}_S \). All that remains is to show that any \( \psi \) which satisfies the hypotheses of the theorem must have fixed points modulo \( \{\phi_n\} \). To this end, let \( s_1, s_2 \) be in \( S \). Then for any \( n = 1, 2, \ldots, \), we have

\[
\| \phi_n(\psi(s_1)) - \phi_n(\psi(s_2)) \| \leq \phi(t_n) \| \psi(s_1) - \psi(s_2) \| \leq \phi(t_n) \| s_1 - s_2 \|.
\]

Since \( \phi(t_n) < 1 \) for every \( n \), we see that \( \phi_n \circ \psi \) is a contraction on \( S \) relative to the norm. But \( S \) is weakly compact, hence norm closed, and, therefore, norm complete. By the Banach contraction principle, \( \psi \circ \phi \) has a fixed point.

Theorem 2 yields as a corollary a fixed point theorem for nonexpansive
self-mappings of compact metric spaces which was first proved by Dotson[3] for nonconvex subsets of Banach spaces.

**Corollary.** Let $X$ be a compact metric space with metric $\rho$, and suppose that there is a continuous function $F : X \times [0, 1] \to X$ such that (i) $F(x, 1) = x$ for every $x$ in $X$, and (ii) there is a self-mapping, $\phi$, of $[0, 1]$ such that for every $x_1, x_2$ in $X$ and for all $t$ in $[0, 1]$ we have

$$\rho(F(x_1, t), F(x_2, t)) \leq \phi(t)\rho(x_1, x_2).$$

Then any nonexpansive self-mapping of $X$ has a fixed point.

**Proof.** It is well known that any metric space can be embedded isometrically in a Banach space (see, e.g., [4, XII, 5.2, p. 286]). Thus, we may consider $X$ to be a compact subset of a Banach space $E$ with metric induced by a norm $\| \cdot \|$. The compactness of $X$ guarantees that the norm topology coincides with the weak topology on $X$ ([4, XI, 2.1, p. 226] or [5, 5.8, p. 141]), and the conclusion follows from Theorem 2.

Following Baron and Matkowski [1], we say that a metric space $(X, \rho)$ is an $S$-space if there exists an $x_0$ in $X$ such that for every $t$ in $[0, 1]$ there is a $\rho$-contractive self-mapping $f_t$ of $X$ for which the inequality $\rho(f_t(x), x) \leq (1 - t)\rho(x_0, x)$ holds for every $x$ in $X$.

**Theorem 3** (Baron and Matkowski [1]). Every nonexpansive self-mapping of a compact $S$-space has a fixed point.

**Proof.** Let $(X, \rho)$ be a compact $S$-space, and let $F$ be a nonexpansive self-mapping of $X$. If the interval $[0, 1]$ is equipped with its usual order, then $\{ f_t : t \in [0, 1] \}$, where $f_t$ is as in the definition of $S$-space, is a net in $X^X$ which converges to $\text{id}_X$. Moreover, $f_t \circ F$ is easily seen to be a contraction on $X$ for each $t$ in $[0, 1]$, so $F$ has fixed points modulo $\{ f_t : t \in [0, 1] \}$.

**BIBLIOGRAPHY**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS