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A FIXED POINT CRITERION FOR COMPACT T_2-SPACES

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ABSTRACT. We prove a fixed point theorem which has as consequences some theorems of W. G. Dotson, Jr., [2], [3] and of K. Baron and J. Matkowski [1].

1. **Introduction and definitions.** In recent articles, W. G. Dotson, Jr., [2], [3] as well as K. Baron and J. Matkowski [1] have extended the Banach contraction principle to nonexpansive self-mappings of certain classes of compact metric spaces. The purpose of this paper is to present an extension of the methods of these authors to certain self-mappings of arbitrary compact T_2-spaces.

Let X be a separated uniform space (see [5] for terminology and notation). A *self-mapping* of X is any function from X into itself. (Note that we do *not* require any continuity of a self-mapping.) The symbol X^X will denote the space of all self-mappings of X equipped with the topology of uniform convergence on X. When $\phi \in X^X$ and $F \subseteq X^X$, we will say that ϕ *has fixed points modulo* F provided that, for each $f \in F$, at least one of the functions $\phi \circ f$ and $f \circ \phi$ has a fixed point.

We shall use the following Proposition, which is an easy consequence of the definitions.

Proposition. *Let* $\{x_\alpha : \alpha \in \mathfrak{A}\}$ *be a net in a uniform space,* X, *which converges to some point* $x_0 \in X$. *If* $\{f_\alpha : \alpha \in \mathfrak{A}\}$ *is a net in* X^X *which converges to a uniformly continuous* f *in* X^X, *then* $\{f_\alpha(x_\alpha) : \alpha \in \mathfrak{A}\}$ *converges to* $f(x_0)$ *in* X.

2. **The main theorem.** We are now ready to give the main theorem. First, recall that a compact T_2-space is, in a canonical way, a separated uniform space, so that the notions of the previous section are meaningful in such a space.

Theorem 1. *Let* X *be a compact* T_2*-space, and let* ϕ *be a continuous self-mapping of* X. *If there is a net* $\{f_\alpha : \alpha \in \mathfrak{A}\}$ *in* X^X *such that* (i) ϕ *has*

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fixed points modulo $\{f_\alpha : \alpha \in \mathfrak{A}\}$, *and* (ii) $\{f_\alpha : \alpha \in \mathfrak{A}\}$ *converges to the identity function*, id_X, *on* X, *then* ϕ *has a fixed point.*

Proof. Condition (ii) requires that id_X be in the closure, in the topology on X^X, of the collection $F = \{f_\alpha : \alpha \in \mathfrak{A}\}$. If we let $F_R = \{f_\alpha \in F : f_\alpha \circ \phi$ has a fixed point$\}$ and $F_L = \{f_\alpha \in F : \phi \circ f_\alpha$ has a fixed point$\}$, then, by (i), $F = F_R \cup F_L$. Thus, we can select a subnet $\{f_\beta : \beta \in \mathfrak{B}\}$ of F which converges to id_X and is entirely contained within F_R or F_L. In the first case, we define $\phi_\beta : X \to X$ for each $\beta \in \mathfrak{B}$ by $\phi_\beta(x) = f_\beta(\phi(x))$; in the second case, we define $\phi_\beta(x) = \phi(f_\beta(x))$. In either case, for each $\beta \in \mathfrak{B}$, there is an x_β in X such that $\phi_\beta(x_\beta) = x_\beta$. Since X is compact, we may assume that $\{x_\beta : \beta \in \mathfrak{B}\}$ converges to some x_0 in X. But $\{\phi_\beta : \beta \in \mathfrak{B}\}$ converges to ϕ in X^X, so application of the Proposition yields that $x_0 = \lim x_\beta = \lim \phi_\beta(x_\beta) = \phi(x_0)$, and x_0 is a fixed point of ϕ.

3. **Consequences.** We now apply Theorem 1 in order to obtain some results due to Dotson and to Baron and Matkowski.

Theorem 2 (Dotson [3]). *Let* S *be a subset of a Banach space* E, *and equip* S *with the relative topology from the weak topology on* E. *Suppose that* S, *so equipped, is compact, and that there exists a continuous function* $F : S \times [0, 1] \to S$ *such that* (i) $F(s, 1) = s$ *for every* s *in* S, *and* (ii) *there is a self-mapping*, ϕ, *of* $]0, 1[$ *such that for every* s_1, s_2 *in* S *and for all* t *in* $]0, 1[$ *we have*

$$\|F(s_1, t) - F(s_2, t)\| \leq \phi(t)\|s_1 - s_2\|.$$

Then any continuous function $\psi : S \to S$ *which is nonexpansive with respect to* $\|\cdot\|$ *has a fixed point.*

Proof. For each $n = 1, 2, \ldots$, let $t_n = n/(n+1)$ and define $\phi_n : S \to S$ by $\phi_n(s) = F(s, t_n)$. It is easy to see that $\{\phi_n\}$ converges in S^S to id_S. All that remains is to show that any ψ which satisfies the hypotheses of the theorem must have fixed points modulo $\{\phi_n\}$. To this end, let s_1, s_2 be in S. Then for any $n = 1, 2, \ldots$, we have

$$\|\phi_n[\psi(s_1)] - \phi_n[\psi(s_2)]\| \leq \phi(t_n)\|\psi(s_1) - \psi(s_2)\| \leq \phi(t_n)\|s_1 - s_2\|.$$

Since $\phi(t_n) < 1$ for every n, we see that $\phi_n \circ \psi$ is a contraction on S relative to the norm. But S is weakly compact, hence norm closed, and, therefore, norm complete. By the Banach contraction principle, $\psi_n \circ \phi$ has a fixed point.

Theorem 2 yields as a corollary a fixed point theorem for nonexpansive self-mappings of compact metric spaces which was first proved by Dotson [3] for nonconvex subsets of Banach spaces.

Corollary. *Let X be a compact metric space with metric ρ, and suppose that there is a continuous function $F: X \times [0, 1] \to X$ such that* (i) $F(x, 1) = x$ *for every x in X, and* (ii) *there is a self-mapping, ϕ, of $]0, 1[$ such that for every x_1, x_2 in X and for all t in $]0, 1[$ we have*

$$\rho(F(x_1, t), F(x_2, t)) \leq \phi(t)\rho(x_1, x_2).$$

Then any nonexpansive self-mapping of X has a fixed point.

Proof. It is well known that any metric space can be embedded isometrically in a Banach space (see, e.g., [4, XII, 5.2, p. 286]). Thus, we may consider X to be a compact subset of a Banach space E with metric induced by a norm $\|\cdot\|$. The compactness of X guarantees that the norm topology coincides with the weak topology on X ([4, XI, 2.1, p. 226] or [5, 5.8, p. 141]), and the conclusion follows from Theorem 2.

Following Baron and Matkowski [1], we say that a metric space (X, ρ) is an *S-space* if there exists an x_0 in X such that for every t in $]0, 1[$ there is a ρ-contractive self-mapping f_t of X for which the inequality $\rho(f_t(x), x) \leq (1 - t)\rho(x_0, x)$ holds for every x in X.

Theorem 3 (Baron and Matkowski [1]). *Every nonexpansive self-mapping of a compact S-space has a fixed point.*

Proof. Let (X, ρ) be a compact S-space, and let F be a nonexpansive self-mapping of X. If the interval $]0, 1[$ is equipped with its usual order, then $\{f_t: t \in]0, 1[\}$, where f_t is as in the definition of S-space, is a net in X^X which converges to id_X. Moreover, $f_t \circ F$ is easily seen to be a contraction on X for each t in $]0, 1[$, so F has fixed points modulo $\{f_t: t \in]0, 1[\}$.

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