THE MONAD SYSTEM OF THE FINEST COMPATIBLE UNIFORM STRUCTURE

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ABSTRACT. The methods of nonstandard analysis are used to study the finest uniform structure compatible with the topology on a given completely regular, Hausdorff space.

Let \((X, \tau)\) be a completely regular, Hausdorff space and let \(J\) be the finest uniform structure on \(X\) which is compatible with \(\tau\). If \(*X\) is the set corresponding to \(X\) in some enlargement, then \(J\) corresponds to a partition \(\{\mu_\mathcal{U}(p) : p \in *X\}\) of \(*X\) in a familiar way [5], [6]. This partition may be described by

\[ q \in \mu_\mathcal{U}(p) \iff *d(p, q) \text{ is infinitesimal for every continuous pseudometric } d \text{ on } (X, \tau). \]

The results in this paper concern the structure of the monad system \(\mu_\mathcal{U}\), especially as it is related to the set \(C(X)\) of continuous, real-valued functions on \((X, \tau)\). Also, it is proved that \(\mu_\mathcal{U}\) is identical to the \(\mu\)-monad system constructed in a quite different way by Wattenberg [8] and some consequences of this fact are discussed.

In general we adopt in this paper the framework for nonstandard analysis which is described in Luxemburg's important paper [5]. Throughout this paper \(\mathcal{M}\) will denote a higher order set-theoretical structure and \(*\mathcal{M}\) will denote an enlargement of \(\mathcal{M}\) which is \(\mathcal{N}\)-enlarging in the sense of [4]. Given a uniform space \((X, \mathcal{U})\) in \(\mathcal{M}\), we let \(\equiv\) be the equivalence relation on \(*X\) which corresponds to \(\mathcal{U}\); that is

\[ p \equiv q \iff (p, q) \in \mathcal{V} \text{ for all } V \in \mathcal{U}. \]

For each \(p \in *X\) the \(\mathcal{U}\)-monad of \(p\), which is denoted by \(\mu_\mathcal{U}(p)\), is the equivalence class of \(p\) under \(\equiv\). A discussion of how the monads \(\mu_\mathcal{U}(p)\) form a basis for the nonstandard theory of uniform spaces may be found in [5] or [6].

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A uniform structure $\widehat{U}$ is generated on $^*X$ by the collection $\{^*V: V \in U\}$ of subsets of $^*X \times ^*X$, as described in [3]. For each $p \in ^*X$, the monad $\mu_U(p)$ is then equal to the intersection of all $\widehat{U}$-neighborhoods of $p$. Therefore, $p \equiv q$ if and only if $p$ and $q$ have exactly the same filter of $\widehat{U}$-neighborhoods. Let $X_0 = \{\mu_U(p): p \in ^*X\}$ and let $U_0$ be the quotient uniformity induced on $X_0$ by $\widehat{U}$. Denote the quotient map from $(^*X, \widehat{U})$ onto $(X_0, U_0)$ by $\pi_U$.

An element $p$ of $^*X$ is called $U$-prenearstandard [5] if for each $V \in U$ there exists $x \in X$ with $(p, x) \in V$. The set of $U$-prenearstandard points is denoted by $\text{pns}_U$. Note that $\text{pns}_U$ is just the $\widehat{U}$-closure in $^*X$ of the set $\{x: x \in X\}$ of standard points in $^*X$. It can be shown that $\pi_U(\text{pns}_U)$ is $U_0$-complete so that if $(X, U)$ is Hausdorff, then the subspace $\pi_U(\text{pns}_U)$ of $(X_0, U_0)$ is the completion of $(X, U)$.

Our purpose in assuming that $^*\mathcal{M}$ is $K_0$-enlarging is to obtain a local description of the $\widehat{U}$-topology on $^*X$ in terms of the $\widehat{U}$-monads, as given in Theorem 1. Recall that $^*\mathcal{M}$ is $K_0$-enlarging if and only if it has the following property described in [5]: if $Y$ is a set in $\mathcal{M}$, $\mathcal{G}$ is a filter on $Y$ and $A$ is an internal subset of $^*Y$, then $A \cap \mu(\mathcal{G}) \neq \emptyset$ if and only if $A \cap ^*W \neq \emptyset$ for every $W \in \mathcal{G}$. (Here $\mu(\mathcal{G})$ is the filter monad of $\mathcal{G}$ defined by $\mu(\mathcal{G}) = \mathcal{G}$. Recall also that for each set $X$ there exists $^*\mathcal{M}$ which is an $K_0$-enlarging extension of some $\mathcal{M}$ containing $X$. (For example, the limit of a sequence of successive enlargements, the first of which has $X$ as an element.)

**Theorem 1.** Let $(X, U)$ be a uniform space in $\mathcal{M}$.

(i) If $A \subseteq ^*X$ is internal, then the interior of $A$ in the $\widehat{U}$-topology is $\{p: \mu_U(p) \subseteq A\}$.

(ii) For $p \in ^*X$, a basis for the $\widehat{U}$-neighborhood filter at $p$ consists of those internal subsets of $^*X$ which contain $\mu_U(p)$.

**Proof.** (i) If $p$ is in the $\widehat{U}$-interior of $A$, then $^*V(p) \subseteq A$ for some $V \in U$ and therefore $\mu_U(p) \subseteq A$. If $p$ is not in the interior of $A$, then $^*V(p)$ intersects $^*X \sim A$ for every $V \in U$. Since $^*\mathcal{M}$ is $K_0$-enlarging it follows that the filter monad of $U$, which is equal to the equivalence relation $\equiv$, intersects $(^*X \sim A) \times \{p\}$. Therefore $\mu_U(p) \not\subseteq A$ if $p$ is not in the interior of $A$.

(ii) This is an immediate consequence of (i) and the fact that the $\widehat{U}$-neighborhood filter at $p$ has a basis $\{^*V(p): V \in U\}$ of internal sets.
The observations in Theorem 1 are useful in the following type of setting: \( \mathcal{U} \) and \( \mathcal{C} \) be two uniform structures on \( X \) and suppose that \( S \) is a subset of \( \ast X \) such that \( \mu_\mathcal{U}(p) \subseteq \mu_\mathcal{C}(p) \) for every \( p \in S \). Then there is a natural function \( \phi \) from \( \pi_\mathcal{U}(S) \) onto \( \pi_\mathcal{C}(S) \) which takes \( \pi_\mathcal{U}(p) \) to \( \pi_\mathcal{C}(p) \) for each \( p \in S \). Theorem 1 implies that the \( \mathcal{U} \)-topology is finer than the \( \mathcal{C} \)-topology when restricted to \( S \). It follows that \( \phi \) is continuous (relative to the \( \mathcal{U}_0 \)-topology on \( \pi_\mathcal{U}(S) \) and the \( \mathcal{C}_0 \)-topology on \( \pi_\mathcal{C}(S) \)). In particular, if \( \mu_\mathcal{U}(p) = \mu_\mathcal{C}(p) \) for every \( p \in S \), then \( \phi \) is a homeomorphism.

Now let \((X, r)\) be a completely regular, Hausdorff space in \( \mathcal{M} \) and let \( J \) be the finest compatible uniform structure on \((X, r)\). In the next result, the difficult part of which is a consequence of Shirota's theorem [2, Theorem 15.21], the relation between \( J \) and \( C(X) \) is explored in terms of non-standard analysis.

**Theorem 2.** For each \( p \in X^* \)

\[ \mu_J(p) \subseteq \{ q : \ast f(p) = \ast f(q) \text{ for all } f \in C(X) \}. \]

If \( p \) is \( J \)-prenearstandard, then equality holds in (1). Moreover, if \((X, r)\) has no closed, discrete subspace of measurable cardinality, then

\[ \text{pns}_J = \{ p : \ast f(p) \text{ is finite for all } f \in C(X) \}. \]

**Proof.** For each \( f \in C(X) \) the function \( |f(x) - f(y)| \) is a continuous pseudometric on \((X, r)\). Therefore \( p \equiv q \) implies \( \ast f(p) = \ast f(q) \) for any \( p, q \in X \) and \( f \in C(X) \). This shows that (1) holds in general.

To prove that equality holds in (1) for prenearstandard points, let \( p \in \text{pns}_J \). If \( q \notin \mu_J(p) \), then for some standard \( \delta > 0 \) and some continuous pseudometric \( d \) on \((X, r)\), \( \ast d(p, q) > \delta \). Since \( p \in \text{pns}_J \), \( \text{pns}_J \) there exists \( x \in X \) which satisfies \( \ast d(p, x) < \delta/3 \). The function \( f(y) = d(y, x) \) is in \( C(X) \), but \( \ast f(p) \) cannot be infinitely close to \( \ast f(q) \) (since otherwise \( \ast d(q, x) < \delta/2 \) and hence \( \ast d(p, q) < \delta \)). Thus \( q \) is also not in the set on the right side of (1).

Now let \( \mathcal{U} \) be the uniform structure on \( X \) generated by the pseudometrics \( |f(x) - f(y)| \) for \( f \in C(X) \). Evidently \( \mathcal{U} \subseteq J \) and therefore \( \text{pns}_J \subseteq \text{pns}_\mathcal{U} \). Moreover, \( \text{pns}_\mathcal{U} \) is equal to the right side of (2) by [5, Theorem 3.15.5]. By the remarks above, a homeomorphism \( \phi \) of \( \pi_J(\text{pns}_J) \) into \( \pi_\mathcal{U}(\text{pns}_\mathcal{U}) \) may be defined by setting \( \phi(\pi_J(p)) \) equal to \( \pi_\mathcal{U}(p) \) for \( p \in \text{pns}_J \). Now \( \pi_J(\text{pns}_J) \) is the completion of \((X, J)\) and \( \pi_\mathcal{U}(\text{pns}_\mathcal{U}) \) is the completion of \((X, \mathcal{U})\). In case the stated assumption holds then Shirota's theorem [2,
Theorem 15.21] implies that both of these completions are real compact. Also they both contain $X$ densely and $\phi$ is the identity map on $X$. Since every function in $C(X)$ extends to a continuous function on $\pi_{\mathcal{U}}(\text{pns}_{\mathcal{U}})$ it follows that $\phi$ maps $\pi_{\mathcal{U}}(\text{pns}_{\mathcal{U}})$ onto $\pi_{\mathcal{U}}(\text{pns}_{\mathcal{U}})$. The first part of the theorem then implies (2).

Next we show that the uniform structure $\mathcal{F}$ is closely related to the monad systems introduced by Wattenberg in [7] and [8]. The most interesting of these systems, which we will call the $\mu$-monad system, is constructed as follows [8, Definition 2.9]: given a completely regular, Hausdorff space $(X, \tau)$ in $\mathcal{M}$, let $\lambda$ be a cardinal number which is greater than $2^{\aleph_0}$ and greater than the cardinality of $X$. Let $S$ be a set of cardinality $\lambda$ and let $R^\lambda$ be the set of real-valued functions $\alpha$ on $S$ whose support $\{s: \alpha(s) \neq 0\}$ is finite. Equip $R^\lambda$ with the metric $d_\lambda$ defined by

$$d_\lambda(\alpha, \beta) = \sup \{|\alpha(s) - \beta(s)|: s \in S\}.$$ 

The $\mu$-monad for $(X, \tau)$ consists of the family $\{\mu(p): p \in \ast X\}$ of subsets of $\ast X$ defined by

$$\mu(p) = \{q: \text{ for each continuous function } f: (X, \tau) \to (\mathbb{R}^\lambda, d_\lambda), \ast d_\lambda(\ast f, p, \ast q) = 0\}.$$ 

The significance of the $\mu$-monad system lies in these facts, proved in [8]:

(i) $\{\mu(p): p \in \ast X\}$ is a partition of $\ast X$ which agrees with the $\tau$-monad partition on $\tau$-nearstandard points;

(ii) if $f: (X_1, \tau_1) \to (X_2, \tau_2)$ is a continuous function between completely regular, Hausdorff spaces in $\mathcal{M}$, then for each $p \in \ast X_1$, $\ast f(\mu(p)) \subseteq \mu(\ast f(p))$. Moreover, the $\mu$-monad system is the same as the metric monad system on metric spaces and is the same as the covering monad system on normal spaces [8, Theorem 2.10], both of which have topologically natural definitions.

Our next result is that the $\mu$-monad on $\ast X$ is identical to the monad $\mu_f^\mathcal{U}$ given by the finest compatible uniform structure $\mathcal{F}$ on $(X, \tau)$. This theorem not only makes a connection between Wattenberg's work and the theory of uniform spaces, but also leads (via his work) to useful descriptions of the $\mu_f^\mathcal{U}$-monads in certain special situations.

**Theorem 3.** If $(X, \tau)$ is a completely regular, Hausdorff space in $\mathcal{M}$ and $\mathcal{F}$ is the finest compatible uniform structure on $(X, \tau)$, then $\mu(p) = \mu_f^\mathcal{U}(p)$ for all $p \in \ast X$. 

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Proof. Let the \( \mu \)-monad for \((X, \tau)\) be constructed as described above. The definition of \( \mu(p) \) for \( p \in \mathcal{P}X \) makes it evident that there is a filter \( \mathcal{U} \) on \( X \times X \) such that the filter monad of \( \mathcal{U} \) equals \( \{(p, q) : q \in \mu(p)\} \). Since this set is an equivalence relation on \( \mathcal{P}X \), it follows that \( \mathcal{U} \) is a uniform structure on \( X \) [5, Theorem 3.9.1]. In addition, \( \mu(p) = \mu_{\mathcal{U}}(p) \) for all \( p \in \mathcal{P}X \). By (i) above, for each \( x \in X \)

\[
\mu_{\mathcal{U}}(x) = \{p : p \text{ is } \tau\text{-nearstandard to } x\} = \mu_{\mathcal{F}}(x).
\]

This implies that the uniform structure \( \mathcal{U} \) is compatible with \( \tau \) and hence \( \mathcal{U} \subseteq \mathcal{F} \).

Now \( \mathcal{F} \) may be described as the coarsest uniform structure on \( X \) for which every continuous function from \((X, \tau)\) into a metric space is actually uniformly continuous. Let \((Y, d)\) be a metric space and \( f : (X, \tau) \to (Y, d) \) a continuous function. To prove that \( f \) is uniformly continuous relative to \( \mathcal{U} \) it suffices to show

\[
(f(\mu(p))) \subseteq \{q : d(f(p), f(q)) = 0\}
\]

for each \( p \in \mathcal{P}X \). Property (ii) of the \( \mu \)-monad system implies that \( f(\mu(p)) \subseteq \mu(f(p)) \). But the \( \mu \)-monad for \((Y, d)\) is the same as the metric monad [8, Theorem 2.10] which is finer than the \( \mathcal{F} \)-monad. That is

\[
\mu(f(p)) \subseteq \{q : d(f(p), f(q)) = 0\}
\]

which yields (3). That is, each continuous function from \((X, \tau)\) into a metric space is uniformly continuous relative to \( \mathcal{U} \), and therefore \( \mathcal{F} \subseteq \mathcal{U} \). This proves \( \mathcal{U} = \mathcal{F} \) so that \( \mu(p) = \mu_{\mathcal{U}}(p) = \mu_{\mathcal{F}}(p) \) for every \( p \in \mathcal{P}X \).

Corollary 1. If \((X, d)\) is a metric space in \( \mathcal{M} \) and \( \mathcal{F} \) is the finest uniform structure compatible with the \( d \)-topology on \( X \), then for \( p \in \mathcal{P}X \)

\[
\mu_{\mathcal{F}}(p) = \{q : d(p, q) < f(p) \text{ for every positive function } f \in C(X)\}.
\]

Proof. Theorem 3 and [8, Theorem 2.10].

The description of the \( \mathcal{F} \)-monads given in Corollary 1 enables us to give a simple proof of a result for subspaces of the Euclidean spaces which is contained in the paper [1] of Corson and Isbell.

Corollary 2. If \((X, \tau)\) is homeomorphic to a subspace of \( \mathbb{R}^n \) for some \( n \geq 1 \) and \( \mathcal{F} \) is the finest compatible uniform structure on \((X, \tau)\), then \( \mathcal{F} \) is the uniform structure on \( X \) generated by the pseudometrics \( |f(x) - f(y)| \) for \( f \in C(X) \). That is, \( \mathcal{F} \) is the unique compatible uniform structure on
which makes every function in $C(X)$ uniformly continuous.

Proof. We assume $X \subseteq \mathbb{R}^n$ and let $\tau$ be the subspace topology so $\tau$ is defined by the metric

$$d(x, y) = \max \{|x_i - y_i| : 1 \leq i \leq n\}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. By Corollary 1 it suffices to show that if $\ast f(p) = \ast f(q)$ for every $f \in C(X)$, then $\ast d(p, q) < \ast g(p)$ for every positive $g \in C(X)$. To prove this, let $p, q$ satisfy the first condition and let $g \in C(X)$ be positive. We will show $\ast d(p, q) < \ast g(p)$, and it may be assumed that $g$ is bounded on $X$. Then

$$1 = \frac{1}{\ast g(q)} \ast g(p)$$

since $1/g(x)$ is in $C(X)$ and $\ast g(q)$ is finite. Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$. Since $x_i/g(x)$ is in $C(X)$, it follows from (4) that

$$\ast p_i / \ast g(p) = q_i / \ast g(p).$$

This shows that $|p_i - q_i| < \ast g(p)$ for $1 \leq i \leq n$ and therefore $\ast d(p, q) < \ast g(p)$, completing the proof.

Now let $(X, \tau)$ be an arbitrary completely regular, Hausdorff space in $\mathbb{M}$ and let $\mathcal{F}$ be the finest compatible uniform structure on $(X, \tau)$. Two elements $p, q$ of $\ast X$ are in the same $\mathcal{F}$-galaxy if for each $V \in \mathcal{F}$ there is a finite sequence $p_0, \ldots, p_k$ in $\ast X$ such that $p_0 = p$, $p_k = q$ and $(p_i, p_{i+1}) \in \ast V$ for each $i = 0, \ldots, k - 1$. In particular this implies that $\ast d(p, q)$ is finite for every continuous pseudometric $d$ on $(X, \tau)$. Therefore, the set $A = \{p : \ast f(p) \text{ is finite for all } f \in C(X)\}$ is a union of $\mathcal{F}$-galaxies, as is $\ast X \sim A$.

If there are no measurable cardinals, then Theorem 2 shows that $\pi_\mathcal{F}(A)$ is the completion of $(X, \mathcal{F})$. Therefore the galaxy structure for points in $A$ reflects in an immediate way the structure of $(X, \tau)$. The next result shows that the galaxy structure for points outside $A$ is trivial. This has the immediate consequence that any connected subset of $X_0 = \pi_\mathcal{F}(\ast X)$ with at least two elements must be contained in $\pi_\mathcal{F}(A)$. That is, $\pi_\mathcal{F}(A)$ is open and closed in $X_0$ and $X_0 \sim \pi_\mathcal{F}(A)$ is totally disconnected (in the $\mathcal{F}_0$-topology on $X_0$).

**Theorem 4.** Let $(X, \tau)$ be a completely regular, Hausdorff space in $\mathbb{M}$ and let $\mathcal{F}$ be the finest compatible uniform structure on $(X, \tau)$. If $p \in \ast X$ and there exists $f \in C(X)$ such that $\ast f(p)$ is infinite, then the $\mathcal{F}$-galaxy containing $p$ is equal to the monad $\mu_\mathcal{F}(p)$. [License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use]
Proof. Suppose $p \in \ast X$ and $f \in C(X)$ and assume that $\ast f(p)$ is infinite. Let $q$ be an element of the same $\mathcal{F}$-galaxy as $p$, so that $\ast d(p, q)$ is finite for every continuous pseudometric $d$ on $(X, r)$ [3, Theorem 4.2].

We may assume $k \geq 1$; let $g = 1/f \in C(X)$ so $0 < g \leq 1$ and $\ast g(p) = 1$.

Given a continuous pseudometric $d$, define

$$d'(x, y) = d(x, y) + |g(x) - g(y)|$$

and

$$V_n = \{(x, y) : d'(x, y) < n^{-1} \cdot g(x)\}$$

for $n \geq 1$. Since $d'$ is a continuous pseudometric on $(X, r)$, the sets $V_n$ are open neighborhoods of the diagonal in $X \times X$. We will show that $V_{n+1} \subseteq (V_n)^{-1}$ and $(V_{4n+1})^2 \subseteq V_n$ for each $n \geq 1$. Suppose $d'(x, y) < (n + 1)^{-1} \cdot g(x)$. Then $|g(x) - g(y)| < (n + 1)^{-1} \cdot g(x)$ so that $(n + 1)^{-1} \cdot g(x) < n^{-1} g(y)$ and $g(y) < (n + 2)(n + 1)^{-1} \cdot g(x)$. It follows that $d'(x, y) < n^{-1} g(y)$, which proves $V_{n+1} \subseteq (V_n)^{-1}$. If also $d'(y, z) < (n + 1)^{-1} \cdot g(y)$, then

$$d'(x, z) < ((n + 1)^{-1} + n^{-1})g(y) < 4n^{-1}g(x).$$

This proves $(V_{4n+1})^2 \subseteq V_n$. It follows from these facts that $\{V_n : n \geq 1\}$ generates a uniform structure on $X$ which defines a topology coarser than $\tau$. Therefore $\{V_n : n \geq 1\}$ is contained in $\mathcal{F}$.

Now since $V_1$ is in $\mathcal{F}$ and since $p$ and $q$ are in the same $\mathcal{F}$-galaxy, there is a finite sequence $p_0, \ldots, p_k$ in $\ast X$ such that $p_0 = p, p_k = q$ and for each $i = 0, \ldots, k - 1$

$$\ast d(p_i, p_{i+1}) + |\ast g(p_i) - \ast g(p_{i+1})| < \ast g(p_i).$$

An inductive argument shows that $\ast g(p_i) = 1$ for each $i = 0, \ldots, k - 1$. Using the triangle inequality for $\ast d$ yields

$$\ast d(p, q) < \sum_{i=0}^{k-1} \ast g(p_i) = 1.$$


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