PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 1, August 1975

ISOLATED POINTS OF THE SPECTRA OF CONSERVATIVE MATRICES

N. K. SHARMA

ABSTRACT. We show that the only possible isolated points in the spectrum of a conservative triangular matrix are its diagonal elements, and that any Hausdorff method corresponding to an absolutely continuous mass function is in the norm closure of the analytic methods.

Let ω represent the space of all real and complex sequences and let c (c_0 or m) represent the subspace of all convergent sequences (null sequences or bounded sequences). Let $B(c)$ represent the complex Banach algebra of bounded operators on c with the usual norm topology. Let $A = (a_{nk})$ be any infinite matrix and let, for x in ω, Ax denote the transformed sequence, provided it exists. Let $c_A = \{x \in \omega: AX \in c\}$. A is called conservative (coercive) if $c_A \supseteq c$ ($c_A \supseteq m$). For the properties of conservative (coercive) matrices and Hausdorff methods the reader might refer to [2]. A matrix A is called triangular if $a_{nk} = 0$ whenever $k > n$ and a triangular matrix A is called a triangle if $a_{nn} \neq 0$ for any n. Δ denotes the Banach algebra of triangular matrices in $B(c)$. If $A \in B(c)$ then the set of complex numbers z such that $A - zI$ is not invertible in $B(c)$ is called the spectrum of A and is denoted by $\sigma(A)$. That $\sigma(A)$ is nonempty and compact is well known [6, p. 418].

In §1 we show that if A is in Δ then the only possible isolated points in $\sigma(A)$ are the a_{nn}'s. This answers Question 2 of [9] negatively. We also show that the spectrum of a multiplicative Hausdorff method is always connected.

In §2 we show that the Hausdorff methods corresponding to absolutely continuous mass functions are indeed in the norm closure of the analytic methods of [3]. We then give an application of this result and end the paper with an open question.

Presented to the Society November 17, 1972; received by the editors August 8, 1973.

AMS (MOS) subject classifications (1970). Primary 40H05, 40G05; Secondary 47B99.

1. We start with

Lemma. *If* $A \in \Delta$ *and* $a_{kk} = 0$ *for some* k*, then* $R(A)$*, range of* A*, is not dense in* c*.*

Proof. We can assume that $a_{kk} = 0$ for some k and that $a_{nn} \neq 0$ for $n < k$. Then for $y = Ax$, it is clear that y_k is a linear combination of y_n's for $n < k$. It is now easy to check that $R(A)$ cannot be dense in c.

Theorem 1. *Let* $A \in \Delta$*. The only possible isolated points in* $\sigma(A)$ *are the* a_{nn}*'s.*

Proof. Let z be an isolated point of $\sigma(A)$. It follows from [6, p. 421] that $\exists$ a nontrivial projection P in Δ which commutes with A and that PA, as an operator on Pc, has spectrum $\{z\}$. It is clear from the proof of the Lemma that if the nth element of the diagonal of P is nonzero, then $a_{nn} \in \sigma(PA)$. This proves the theorem.

Corollary 1. *There does not exist a triangle in* Δ *with zero as an isolated point of its spectrum.*

This answers Question 2 of [9] negatively.

Corollary 2. *If* $A \in \Delta$ *and* $a_{nn} = a$ *for every* n*, then* $\sigma(A)$ *is connected.*

Proof. Suppose $\sigma(A)$ is not connected. $\exists$ nonempty open sets U and V such that $U \cap V = \emptyset$ and $\sigma(A) \subseteq U \cup V$. Let $a \in U$. Then we have a nontrivial projection P such that PA, as an operator on Pc, has spectrum $V \cap \sigma(A)$. But $\sigma(PA)$ must contain some a_{nn} and hence the point a which contradicts $U \cap V = \emptyset$.

Remark 1. Corollary 3 shows that $a_{nn} = a$ for every n is not a necessary condition for spectrum to be connected.

Theorem 2. *Let* $P \in \Delta$ *and* P *Hausdorff. Then* $P^n = P$ $(n \geq 2)$ *if and only if* $P \in \{0, I, \pm H_0, \pm(I - H_0)\}$*, where* H_0 *is the Hausdorff method* $\{h_{nk}\}$ *with* $h_{n0} = 1$ *for every* n *and* $h_{nk} = 0$ *otherwise.*

Proof. Let $P \neq 0, I$ and let $Q = P^{n-1}$. Then $Q^2 = P^{2n-2} = P^n \cdot P^{n-2} = P^{n-1} = Q$. Therefore $\sigma(Q) = \{0, 1\}$. But in view of the spectral mapping theorem [6, p. 432], $Q = P^{n-1}$ implies that $\sigma(Q) = [\sigma(P)]^{n-1}$. Hence $\sigma(P)$ must be finite. It follows from [9, Theorem 2] that $P \in \{\pm H_0, \pm(I - H_0)\}$.

Corollary 3. *Let* H *be a Hausdorff method. Then* $\sigma(H)$ *consists of at* *most two components, one of which is a singleton set* $\{\mu_0\}$, *where* μ_0 *is the first diagonal element of* H.

Proof. Following the proof of Corollary 2 we find that if $\sigma(H)$ is disconnected, then the corresponding projection P can be chosen to equal H_0. But $H_0 H = \mu_0 H_0$, and $\mu_0 H_0$, as an operator on $H_0 c$, has spectrum $\{\mu_0\}$.

Theorem 3. *Let* $A \in \Delta$ *and suppose* a_{nn}*'s are all distinct. If* z *is an isolated point in* $\sigma(A)$ *then* z *belongs to the point spectrum of* A.

Proof. Theorem 1 tells us that $z = a_{rr}$ for some r. If $P = \{p_{nk}\}$ is the corresponding projection, then in view of the Lemma it follows that $p_{rr} = 1$ and $p_{kk} = 0$ for $k \neq r$. Direct computation shows that $p_{nk} = 0$ for $k > r$. Hence Pc is finite dimensional and therefore PA, as an operator on Pc, has only point spectrum. Consequently z belongs to the point spectrum of A.

Remark 2. It follows from Pitt's theorem [5] that if H is a multiplicative Hausdorff method corresponding to a mass function of regular bounded variation then spectrum of H is connected. It follows from Corollary 3 that if H is any multiplicative Hausdorff method then spectrum of H is connected.

2. It has been observed in [9, following Facts 1 and 2] that if H is any conservative Hausdorff method then there exists f, analytic in $D = \{z: |z - \tfrac{1}{2}| < \tfrac{1}{2}\}$ such that $H = f(M)$, where M is the Cesàro method of order 1, and also that $\sigma(H) \supseteq \overline{f(D)}$. One would like to know when $\sigma(H) = \overline{f(D)}$. It is well known that $\mathfrak{U}$, the set of all conservative Hausdorff methods, is a commutative Banach algebra with identity and as such $z \in \sigma(H)$ if and only if there exists a multiplicative linear functional χ on $\mathfrak{U}$ such that $\chi(H) = z$. If $\chi(H) = \chi(f(M)) = f(\chi(M))$, it is clear that $\sigma(H) = \overline{f(D)}$. In the following we investigate this phenomenon.

Theorem 4. *Let* H *be a conservative Hausdorff method which corresponds to an absolutely continuous mass function* ψ. *Then there exists* $\{f_n\}$ *analytic in* $\overline{D}$ *such that* $\{f_n(M)\}$ *converges to* H *in norm.*

Proof. Let $H = \{h_{nk}\}$. Then

$$h_{nk} = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} \psi'(t)\, dt.$$

Since $\int_0^1 |\psi'(t)|\, dt < \infty$, it follows from [10, p. 167] that there exist polynomials in $\log t$ such that

$$\int_0^1 |\psi'(t) - p_n(\log t)|\, dt \to 0 \quad \text{as } n \to \infty.$$

Let $g_n(z) = \int_0^1 t^{1/z-1}(\log t)^n\, dt$, for $z \in D$. Then $g_n(z) = (-1)^n n! z^{n+1}$. Clearly $g_n(z)$ is analytic in $\overline{D}$ and $g_n(M) = (-1)^n n! M^{n+1}$. Thus nonconstant polynomials in $M, M^2, M^3, \ldots, M^n, \ldots$ are norm dense in the set of Hausdorff methods corresponding to absolutely continuous mass functions.

Corollary 4. *If H corresponds to an absolutely continuous mass function then* $\sigma(H) = \overline{f(D)}$.

Proof. It follows from above that $\exists \{f_n\}$, f_n analytic in $\overline{D}$, such that $f_n(M) \to H$. Also it is easy to check that for any multiplicative linear functional χ, $\chi(f_n(M)) = f_n(\chi(M))$. Observe that

$$\|f(M)\| \geq \sup\{|f(z)|: z \in \overline{D} - \{0\}\}.$$

Hence $f_n(M) \to H$ implies that $f_n(z) \to f(z)$ uniformly on $\overline{D} - \{0\}$. If $\chi(M) \neq 0$, then it follows that $\chi(f(M)) = f(\chi(M))$. If $\chi(M) = 0$ then $\chi(f(M)) = \lim_{n\to\infty} f_n(0) = 0$. f may not be defined at 0 but it is easy to check that $\lim_{z\to 0; z\in D} f(z) = 0$. Hence $\sigma(H) = \overline{f(D)}$.

Applications. (1) It is clear that Corollary 4 extends to discrete Hausdorff operators on l^p, $p > 1$. In particular one gets Theorem 2 of [4].

(2) Let ψ be absolutely continuous in $[0, 1]$ and $\mu(z) = \int_0^1 t^z \psi'(t)\, dt$. Then it follows from Corollary 4 that the Hausdorff method H corresponding to the moment generating sequence $\{\mu(n)\}$ is equivalent to convergence if and only if $\inf\{|\mu(z)|: \operatorname{Re} z \geq 0\} \geq \beta > 0$. This is a special case of Pitt's theorem [5, Theorem 7].

Remark 3. If $H_1 = f(M)$ and $H_2 = g(M)$ and χ is any linear multiplicative functional on $\mathfrak{A}$ such that $\chi(H_1) = f(z_1)$ and $\chi(H_2) = g(z_2)$, then it is not necessary that $f(z_1) = f(z_2)$ or $g(z_1) = g(z_2)$. For example we can take $H_1 = M$ and $H_2 = H_0$ and

$$\chi(A) = \lim_{n\to\infty} \sum_k a_{nk} - \sum_k \left(\lim_{n\to\infty} a_{nk}\right)$$

for any conservative matrix A. Then it is well known that χ is a multiplicative linear functional and $\chi(H_1) = 1$, $\chi(H_2) = 0$. Here $f(z) = z$ and $g(0) = 1$ but $g(z) = 0$ otherwise. In case $\chi(M) = z_0$, $z_0 \in D$, then one can prove that $\chi(f(M)) = f(\chi(M))$ [*Proof.* The convergence domain of $f(M) - f(z_0)I$ contains convergence domain of $M - z_0 I$ (this is a consequence of Agnew's theorem [1] and Rhoades' theorem [7]). Then in view of Theorem 2 of Rogosinski [8] one gets $f(M) - f(z_0)I = (M - z_0 I)h(M)$ for some conservative Hausdorff method $h(M)$. Consequently $\chi(f(M)) = f(z_0) = f(\chi(M))$. This proof also shows that $\chi(M) = z_0$ and $z_0 \in D$ if and only if χ is the point evaluation at $z_0 \in D$.] The example in the beginning of the paragraph shows that this is no longer true if $z_0 \in \partial D$. There is one open problem: If $\sigma(f(M)) = \overline{f(D)}$ and $\sigma(g(M)) = \overline{g(D)}$, is it true that

$$\sigma(f(M) + g(M)) = \overline{(f + g)(D)}?$$

We take this opportunity to thank Professor Rhoades for pointing out an error in the earlier proof of Theorem 3. We also wish to thank the referee of this paper for his valuable remarks.

REFERENCES

1. R. P. Agnew, *On Hurwitz-Silverman-Hausdorff methods of summability*, Tôhoku Math. J. 49 (1942), 1–14. MR 7, 433.

2. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949. MR **11**, 25.

3. W. A. Hurwitz and L. L. Silverman, *On the consistency and equivalence of certain definitions of summability*, Trans. Amer. Math. Soc. **18** (1917), 1–20.

4. Gerald Leibowitz, *Discrete Hausdorff transformations*, Proc. Amer. Math. Soc. **38** (1973), 541–544. MR **47** #4057.

5. H. R. Pitt, *Mercerian theorems*, Proc. Cambridge Philos. Soc. 34 (1938), 510–520.

6. F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Akad. Kiadó, Budapest, 1953; English transl., Ungar, New York, 1955. MR **15**, 132; **17**, 175.

7. B. E. Rhoades, *Size of convergence domains for known Hausdorff prime matrices*, J. Math. Anal. Appl. 19 (1967), 457–468. MR 35 #3316.

8. W. W. Rogosinski, *On Hausdorff's method of summability*, Proc. Cambridge Philos. Soc. **38** (1942), 166–192. MR 3, 296.

9. N. K. Sharma, *Spectra of conservative matrices*, Proc. Amer. Math. Soc. **35** (1972), 515–518. MR **46** #5891.

10. ———, *Hausdorff operators*, Acta Sci. Math. (Szeged) **35** (1973), 165–167.

DEPARTMENT OF MATHEMATICS AND STATISTICS, PAHLAVI UNIVERSITY, SHIRAZ, IRAN