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PRESERVATION OF UNIFORM ASYMPTOTIC STABILITY UNDER PERTURBATIONS

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ABSTRACT. Suppose the trivial solution of the interval initial value problem for a linear, convolution, Volterra integrodifferential equation is uniformly asymptotically stable. If the kernel in this equation is integrable, then it is shown that this stability is preserved under perturbations which are nonantisipative and of order two or more.

Our purpose here is to show that if the trivial solution of the initial value problem

$$\text{(L)} \qquad \begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t-s)x(s)\,ds \qquad (t \geq \tau), \\ x(t) &= f(t) \quad \text{on } 0 \leq t \leq \tau \end{aligned}$$

is uniformly asymptotically stable in the sense of [3, p. 489], then this stability is preserved under higher order perturbation. Here $\tau \geq 0$, $f \in C[0, \infty)$, A and $B(t)$ are matrices and $x(t)$ and $f(t)$ are n-vector valued functions. Specifically we prove

Theorem 1. *Suppose $B \in L^1(0, \infty)$ and suppose that the trivial solution of* (L) *is uniformly asymptotically stable. Let $p(x)$ be a continuous function such that $p(0) = 0$ and for any $\alpha > 0$ there exists $\beta > 0$ such that*

$$\text{(1)} \qquad |p(x) - p(y)| \leq \alpha|x - y| \qquad (|x|, |y| \leq \beta).$$

Then the trivial solution of the perturbed initial value problem

$$\text{(P)} \qquad z'(t) = Az(t) + \int_0^t B(t-s)z(s)\,ds + p(z(t))$$

for $t \geq \tau$, $z(t) = f(t)$ on $[0, \tau]$, is also uniformly asymptotically stable.

The initial value problem (L) is quite natural from the point of view of differential equations and delay differential equations. From the usual

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point of view in Volterra problems it would be more natural to replace the the solution $x(t)$ of (L) by $y(t) = x(t + \tau)$. Then $y(t)$ would solve (2) below with $p(y) \equiv 0$, $y_0 = f(\tau)$ and F from the set

$$N = \left\{ F(t) = \int_0^\tau B(t + \tau - s) f(s)\, ds;\ f \in C[0, \tau],\ \tau > 0 \right\}.$$

Stability theory for (2) has been developed in [1]. Although the stability part of the theory in [1] can be used here, the asymptotic stability results are too different to be useful. There are two basic problems. If F and f are related as in N above, then f small implies F small, but not conversely. Moreover, although a uniformly bounded set of functions F is not compact, a set of functions F corresponding to a uniformly bounded set of f's is compact in the uniform norm on $[0, \infty)$. This compactness and the boundedness of the f's are needed to prove the uniformity of the asymptotic stability.

It should be noted that if N is replaced by its closed span, then uniform stability plus asymptotic stability in the sense of [1] is equivalent to uniform asymptotic stability of (L) in the sense of [3]. The proof of this fact is rather complicated; see [5, Theorem 5].

Consider the related problem

$$y'(t) = F(t) + Ay(t) + \int_0^t B(t - s) y(s)\, ds + p(y(t)) \tag{2}$$

for $t \geq 0$ with $y(0) = y_0$. Results in [1] or [4] immediately yield

Theorem 2. *If $B \in L^1(0, \infty)$, $p(y)$ is continuous with $p(y) = o(|y|)$ as $|y| \to 0$, and (L) is uniformly asymptotically stable, then for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|y_0| + |F(t)| \leq \delta$ for all $t \geq 0$ then the solution of (2) exists and satisfies $|y(t)| \leq \epsilon$ for all $t \geq 0$. Moreover if $F(+\infty) = 0$, then $y(+\infty)$ exists and equals zero.*

Bownds and Cushing [1] note that this theorem implies

Corollary 1. *Under the hypothesis of Theorem 2 above, the trivial solution of (P) is uniformly stable in the sense of [3, Definition 4].*

Proof. Fix τ and f and then put $y(t) = z(t + \tau, \tau, f)$ where $z(t, \tau, f)$ solves (P). Then $y(t)$ solves (2) with $y_0 = f(\tau)$ and

$$F(t) = \int_{-\tau}^0 B(t - s) f(s + \tau)\, ds.$$

Since

$$|y_0| + |F(t)| \leq \left(1 + \int_0^\infty |B(s)|\, ds\right) \max\{|f(t)|:\ 0 \leq t \leq \tau\},$$

then Theorem 2 implies the conclusion.

Proof of Theorem 1. By the results in [2] or [4] it follows that the resolvent $R(t)$ associated with (L) is of class $L^1(0, \infty) \cap C[0, \infty)$ and $|R(+\infty)| = 0$. Let $M_1 = \sup\{|R(t)|: t \geq 0\}$ and $M_2 = \int_0^\infty |R(t)|\, dt$. Pick α such that $\alpha M_2 < 1$. Then there exists $\beta > 0$ such that (1) is true whenever $|x|, |y| \leq \beta$. By Theorem 2 and Corollary 1 there exists $\delta > 0$ such that: (a) if $\tau \geq 0$ and $|f(t)| \leq \delta$ on $[0, \tau]$, then $|z(t)| \leq \beta$ for all $t \geq \tau$; and (b) if $|y_0| + |F(t)| \leq \delta(1 + M_2)$ for all $t \geq 0$, then $|y(t)| \leq \beta$ for all $t \geq 0$.

In order to complete the proof it is sufficient to show that for any $\tau \geq 0$ and f if $z(t, \tau, f)$ is the corresponding solution of (P), then $z(t + \tau, \tau, f) \to 0$ as $t \to \infty$ uniformly on the set $\{(\tau, f): |f(t)| \leq \delta$ and $0 \leq t \leq \tau\}$. For a contradiction assume that this is false. Then there exists $\epsilon > 0$ and sequences f_n, τ_n, and $t_n \to \infty$ with $|f_n(t)| \leq \delta$ on $[0, \tau_n]$ but $|z(t_n + \tau_n, \tau_n, f_n)| \geq \epsilon$. Let $y_n(t) = z(t + \tau_n, \tau_n, f_n)$ so that y_n solves (2) with $y_n(0) = f_n(\tau_n)$ and $F = F_n$ where

$$F_n(t) = \int_{-\tau_n}^0 B(t - s) f_n(s + \tau_n)\, ds.$$

It is easy to see that the sequence $\{F_n\}$ is uniformly bounded by $M_2\delta$. The sequence is also equicontinuous since

$$\begin{aligned} |F_n(t + h) - F_n(t)| &= \left| \int_{-\tau_n}^0 \{B(t + h - s) - B(t - s)\} f(s + \tau_n)\, ds \right| \\ &\leq \int_{-\infty}^0 |B(t + h - s) - B(t - s)|\, \delta\, ds. \end{aligned}$$

For any $t \geq 0$ the last expression tends to zero as $h \to 0$ with $t + h \geq 0$. Moreover $F_n(t) \to 0$ as $t \to \infty$ uniformly in n since

$$|F_n(t)| \leq \int_{-\tau_n}^0 |B(t - s)|\, \delta\, ds \leq \delta \int_t^\infty |B(s)|\, ds.$$

Thus there exists a function F such that

$$\epsilon_n = \sup\{|F_n(t) - F(t)|: t \geq 0\} \to 0 \quad \text{as } n \to \infty.$$

Clearly $F \in C[0, \infty)$, $F(\infty) = 0$ exists and $|F(t)| \leq \delta M_2$. Since $|f_n(\tau_n)| \leq \delta$ for all n, by possibly taking a subsequence, we can assume that $|f_n(\tau_n) - y_0| \leq \epsilon_n$ for all n and for some y_0. Let $y(t)$ be the solution of (2) for this F and this y_0. Since $|y_0| + |F(t)| \leq \delta + M_2\delta$ for all $t \geq 0$, then by the choice of δ, $|y(t)| \leq \beta$ for all $t \geq 0$ and $y(\infty) = 0$.

Let $w_n(t) = y_n(t) - y(t)$ for all $t \geq 0$ so that

$$\begin{aligned} w_n'(t) = \{F_n(t) - F(t)\} + A w_n(t) + \int_0^t B(t - s) w_n(s)\, ds \\ + \{p(w_n(t) + y(t)) - p(y(t))\}. \end{aligned} \tag{3}$$

Since $R(t)$ is the resolvent of (L), then (3) can be rewritten in the equivalent form

$$w_n(t) = R(t)\{f_n(\tau_n) - y_0\}$$
$$+ \int_0^t R(t-s)\{F_n(s) - F(s) + p(w_n(s) + y(s)) - p(y(s))\}\,ds.$$

Estimate as follows:

$$|w_n(t)| \le |R(t)|\,|f_n(\tau_n) - y_0| + \int_0^t |R(t-s)|\{|F_n(s) - F(s)| + \alpha|w_n(s)|\}\,ds$$
$$\le M_1\epsilon_n + \int_0^t |R(t-s)|\{\epsilon_n + \alpha\|w_n\|_t\}\,ds$$
$$\le M_1\epsilon_n + M_2\{\epsilon_n + \alpha\|w_n\|_t\},$$

where $\|w_n\|_t = \max\{|w_n(s)| = 0 \le s \le t\}$. It follows that

$$\|w_n\|_t \le \{M_1 + M_2\}\epsilon_n + \alpha M_2\|w_n\|_t,$$

and

$$\|w_n\|_t \le (1 - \alpha M_2)^{-1}(M_1 + M_2)\epsilon_n \longrightarrow 0$$

as $n \to \infty$ uniformly for $t \ge 0$. In particular there exists an N such that if $n \ge N$ then $|y(t_n)| < \epsilon/2$ and $|w_n(t_n)| = |y_n(t_n) - y(t_n)| < \epsilon/2$. This is a contradiction because

$$\epsilon \le |y_n(t_n)| \le |y_n(t_n) - y(t_n)| + |y(t_n)| < \epsilon.$$

This completes the proof of Theorem 1.

Note that the same argument works if the perturbation p in Theorem 1 is any smooth, nonantisipative functional of higher order. For example if $p_j \in C^1$, $p_j(0) = p_j'(0) = 0$ for $j = 1, 2$ and $D(t) \in L^1(0, \infty)$, then

$$p(x) = p_1(x(t)) + \int_0^t D(t-s)p_2(x(s))\,ds$$

will do.

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