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CHARACTERIZATIONS OF BOUNDED MEAN OSCILLATION

STEPHEN JAY BERMAN

ABSTRACT. Recall that an integrable function f on a cube Q_0 in $\mathbf{R}^n$ is said to be of bounded mean oscillation if there is a constant K such that for every parallel subcube Q of Q_0 there exists a constant a_Q such that $\int_Q |f-a_Q| \le K|Q|$, where $|Q|$ denotes the volume of Q. We prove here that if there is an integer d and a constant K such that for every parallel subcube Q of Q_0 there exists a polynomial p_Q of degree $\le d$ such that $\int_Q |f-p_Q| \le K|Q|$, then f is of bounded mean oscillation.

Let $\mathbf{R}^n$ denote n-dimensional Euclidean space, let Q_0 be a finite cube in $\mathbf{R}^n$ (all cubes are assumed to have sides parallel to the axes), and let $L^1(Q_0)$ denote the space of complex-valued integrable functions on Q. As defined by John and Nirenberg in [2], a function $f \in L^1(Q_0)$ is of bounded mean oscillation if there is a constant K such that for every subcube $Q \subset Q_0$ there is a constant a_Q such that $\int_Q |f-a_Q| \le K|Q|$, where $|Q|$ denotes the volume of Q. In this note we prove that if in the definition above, the constant a_Q is replaced by a polynomial p_Q of bounded degree, then f is still of bounded mean oscillation. Precisely, we show the following

Theorem. *Let Q_0 be a finite cube in $\mathbf{R}^n$, let $f \in L^1(Q_0)$, and let d be a fixed positive integer. Suppose there exists a constant K such that for every cube $Q \subset Q_0$ there exists a polynomial p_Q of degree $\le d$ such that $\int_Q |f-p_Q| \le K|Q|$. Then f is of bounded mean oscillation.*

The proof depends upon a series of lemmas about polynomials.

We will use the following notation: For any $m > 0$, let C_m be the cube $\{x \in \mathbf{R}^n: 0 \le x_j \le m \text{ for } j = 1, 2, \ldots, n\}$. For any polynomial p and cube Q, let $\sup(p, Q) = \sup\{|p(x)|: x \in Q\}$ and $\operatorname{osc}(p, Q) = \sup\{|p(x) - p(y)|: x, y \in Q\}$. Let D_j denote differentiation with respect to x_j.

Lemma 1. *There is a constant α (depending only on n and d) such that*

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$$\frac{\alpha}{|Q|}\int_Q |p| \geq \sup(p, Q)$$

for every cube Q *in* $\mathbf{R}^n$ *and every polynomial* p *of degree* $\leq d$.

Proof. Because the $L^1(C_1)$ and supremum norms are equivalent on the finite-dimensional space of polynomials of degree $\leq d$, there is a constant α such that $\alpha\int_{C_1}|p| \geq \sup(p, C_1)$ if $\deg(p) \leq d$. If $Q = \{x \in \mathbf{R}^n\colon a_j \leq x_j \leq a_j + s$ for all $j\}$, then let ϕ map C_1 to Q linearly, that is, if $\phi(x) = y$ then $y_j = a_j + sx_j$ for all j. Then $\alpha\int_{C_1}|p\circ\phi| \geq \sup(p\circ\phi, C_1) = \sup(p, Q)$, which is what we want since $\int_{C_1}|p\circ\phi| = |Q|^{-1}\int_Q|p|$.

Lemma 2. *There exists a constant* β (*depending only on* n *and* d) *such that*

$$\sup(p, C_m) \geq \beta m \sum_{j=1}^{n} \sup(D_j p, C_m) \tag{1}$$

for every $m > 0$ *and every polynomial* p *of degree* $\leq d$.

Proof. On the vector space of polynomials q such that $\deg(q) \leq d$ and $q(0) = 0$, the norms $\sup(q, C_1)$ and $\Sigma_{j=1}^n \sup(D_j q, C_1)$ are equivalent. Therefore there exists a constant β' such that

$$\sup(q, C_1) \geq \beta' \sum_{j=1}^{n} \sup(D_j q, C_1) \tag{2}$$

if $q(0) = 0$ and $\deg(q) \leq d$.

Given any $m > 0$ and any p such that $p(0) = 0$ and $\deg(p) \leq d$, define $\phi\colon C_1 \to C_m$ so that if $\phi(x) = y$ then $y_j = mx_j$, $j = 1, \ldots, n$. Let $q = p\circ\phi$. Using this q in (2) gives (1) with β' in place of β. So (1) holds if $p(0) = 0$ (with β' in place of β). If $p(0) \neq 0$, apply (1) to $p - p(0)$ and use the fact that $\sup(p - p(0), C_m) \leq 2\sup(p, C_m)$ to get (1) for all p such that $\deg(p) \leq d$ (with $\beta = \beta'/2$).

Lemma 3. *There exists a constant* γ (*depending only on* n *and* d) *such that*

$$\operatorname{osc}(p, Q_m) \geq \gamma m \operatorname{osc}(p, Q)$$

for every polynomial p *of degree* $< d$ *and every pair of cubes* Q_m, Q *such that* $Q \subset Q_m$ *and* $|Q_m| = m^n|Q|$.

Proof. Suppose first that $Q = C_1$ and $Q_m = C_m$. We may suppose without loss of generality that $p(0) = 0$. By Lemma 2 we have

(3) $\operatorname{osc}(p, C_m) \geq \sup(p, C_m) \geq \beta m \sum_{j=1}^{n} \sup(D_j p, C_m) \geq \beta m \sum_{j=1}^{n} \sup(D_j p, C_1).$

But if $x \in C_1$, then

$$p(x) = \int_0^1 \frac{d}{dt} p(tx)\, dt = \int_0^1 \sum_{j=1}^{n} D_j p(tx)\, x_j\, dt$$

so that

$$\sum_{j=1}^{n} \sup(D_j, C_1) \geq \sup(p, C_1). \tag{4}$$

Also $\sup(p, C_1) \geq 2^{-1}\operatorname{osc}(p, C_1)$, and this, together with (3) and (4), gives

$$\operatorname{osc}(p, C_m) \geq 2^{-1}\beta m \operatorname{osc}(p, C_1). \tag{5}$$

Now suppose that Q and Q_m are arbitrary cubes satisfying the conditions of the lemma. Geometrically it is clear that we can find a cube Q_k such that $Q \subset Q_k \subset Q_m$, $|Q_k| = k^n|Q|$, $k \geq m/2$, and Q is in a corner of Q_k. We can find a linear change of variable ϕ: $C_k \longrightarrow Q_k$ such that $\phi(C_1) = Q$. Then (5) implies

$$\begin{aligned}\operatorname{osc}(p, Q_m) \geq \operatorname{osc}(p, Q_k) &= \operatorname{osc}(p \circ \phi, C_k) \geq 2^{-1}\beta k \operatorname{osc}(p \circ \phi, C_1)\\ &= 2^{-1}\beta k \operatorname{osc}(p, Q) \geq 4^{-1}\beta m \operatorname{osc}(p, Q).\end{aligned}$$

Therefore we may take $\gamma = \beta/4$.

Proof of the Theorem. We will prove that there exists a constant M such that $\operatorname{osc}(p_Q, Q) \leq M$ for all $Q \subset Q_0$. Once we have this, let $a_Q = |Q|^{-1}\int_Q p_Q$. Then

$$\begin{aligned}\int_Q |f - a_Q| &\leq \int_Q |f - p_Q| + \int_Q |p_Q - a_Q|\\ &\leq K|Q| + \int_Q \left| |Q|^{-1}\int_Q p(y)\, dx - |Q|^{-1}\int_Q p(x)\, dx \right| dy\\ &\leq K|Q| + \int_Q |Q|^{-1}\int_Q \operatorname{osc}(p, Q) \leq K|Q| + M|Q|,\end{aligned}$$

so that f is of bounded mean oscillation as wanted.

To show that such M exists, we suppose the contrary. For each $M > 0$, let $F(M) = \sup\{|Q| : \operatorname{osc}(p_Q, Q) \geq M\}$. Then $F(M) \leq |Q_0|$ and $F(M)$ decreases as $M \longrightarrow \infty$.

We claim that $F(M) \longrightarrow 0$ as $M \longrightarrow \infty$. If not, then there exists $r > 0$ such that for every $M > 0$ there exists a cube Q such that $|Q| > r$ and $\operatorname{osc}(p_Q, Q)$

$\geq M$. But then $\sup(p_Q, Q) \geq M/2$ and Lemma 1 implies $\int_Q |p_Q| > (2\alpha)^{-1}M|Q|$. Therefore

$$\int_Q |f| \geq \int_Q |p_Q| - \int_Q |f - p_Q| \geq ((2\alpha)^{-1}M - K)r.$$

Since M is arbitrary, this contradicts the integrability of f. So $F(M) \longrightarrow 0$ as $M \longrightarrow \infty$ as wanted.

Now fix $M > 0$. Let $\epsilon > 0$. Choose a cube Q so that $|Q| > F(M) - \epsilon$ and $\operatorname{osc}(p_Q, Q) \geq M$. Consider a cube Q_m such that $Q \subset Q_m \subset Q_0$ and $|Q_m| = m^n|Q|$, for a value of m to be specified later. Then, writing p_m for p_{Q_m}, we have $\int_Q |f - p_Q| \leq K|Q|$ and $\int_Q |f - p_m| \leq \int_{Q_m} |f - p_m| \leq K|Q_m| = Km^n|Q|$, so that

$$\int_Q |p_Q - p_m| \leq K(1 + m^n)|Q|.$$

By Lemma 1, $\sup(p_Q - p_m, Q) \leq \alpha K(1 + m^n)$. Since $\operatorname{osc}(p_Q, Q) \geq M$, we have $\operatorname{osc}(p_m, Q) \geq M - 2\alpha K(1 + m^n)$. By Lemma 3, $\operatorname{osc}(p_m, Q_m) \geq \gamma m(M - 2\alpha K(1 + m^n))$. By definition of F, we have

$$F(\gamma mM - 2\alpha\gamma mK(1 + m^n)) \geq |Q_m| = m^n|Q| \geq m^n(F(M) - \epsilon).$$

Letting $\epsilon \longrightarrow 0$, we get

$$F(\gamma mM - 2\alpha\gamma mK(1 + m^n)) \geq m^n F(M). \tag{6}$$

Now if M is sufficiently large, then there exists m_M such that $\gamma^{-1} \leq m_M \leq 2\gamma^{-1}$ and the arguments of F in (6) are equal, that is,

$$\gamma mM - 2\alpha\gamma Km(1 + m^n) = M \qquad (m = m_M). \tag{7}$$

Assume the existence of such m_M for the moment. Then because m_M is bounded but $F(M) \longrightarrow 0$ as $M \longrightarrow \infty$, it follows that when M is large and $m = m_M$ a cube Q_m can be chosen so that $Q \subset Q_m \subset Q_0$ and $|Q_m| = m^n|Q|$. Therefore (6) holds for M large and $m = m_M$, that is, $F(M) \geq (m_M)^n F(M)$ for large M, an impossibility.

So to complete the proof we merely have to show the existence of a solution m_M of (7) between γ^{-1} and $2\gamma^{-1}$ when M is large. But clearly $\gamma mM - 2\alpha\gamma Km(1 + m^n) < M$ when $m = \gamma^{-1}$ for any M, while $\gamma mM - 2\alpha\gamma Km(1 + m^n) > M$ if $m = 2\gamma^{-1}$ and $M > 4\alpha K(1 + 2^n\gamma^{-n})$, so we are done.

Remark 1. When $d = 0$, the natural choice of p_Q is the average value of f on Q. In fact, if $\int_Q |f - a_Q| \leq K|Q|$ for some constant a_Q and if $p_Q = |Q|^{-1}\int_Q f$, then $\int_Q |f - p_Q| \leq 2K|Q|$, as mentioned in [2].

When $n = d = 1$ and $Q = \{x: a - h < x < a + h\}$, then a natural choice for

p_Q is $p_Q(x) = d_1(x-a) + d_2$, where

$$d_1 = h^{-2}\left(\int_a^{a+h} f - \int_{a-h}^{a} f\right) \quad \text{and} \quad d_2 = (2h)^{-1}\int_{a-h}^{a+h} f.$$

The coefficient d_1 is an estimate of $f'(a)$ using averages of f in place of f in the difference quotient $(f(a+h/2) - f(a-h/2))/h$. We claim that if $q(x) = c_1(x-a) + c_2$ is any linear polynomial such that $\int_Q |f-q| \le K|Q|$, then $\int_Q |f - p_Q| \le 3K|Q|$. For

$$|d_1 - c_1| = \left| h^{-2}\int_a^{a+h}(f-q) - h^{-2}\int_{a-h}^{a}(f-q)\right| \le h^{-2}\int_Q |f-q| \le h^{-2}K|Q|$$

and

$$|d_2 - c_2| = \left|(2h)^{-1}\int_{a-h}^{a+h}(f-q)\right| \le K$$

so that

$$\begin{aligned}\int_Q |f-p_Q| &\le \int_Q |f-q| + \int_Q |q-p_Q| \\ &\le K|Q| + |c_1 - d_1|\int_{a-h}^{a+h} |x-a|\,dx + |c_2 - d_2|Q| \\ &\le K|Q| + K|Q| + K|Q| = 3K|Q|\end{aligned}$$

as wanted.

There is a theorem of Bernstein which says that if $f^{(d+1)}(x) > 0$ for $-1 < x < 1$, then the polynomial of degree $\le d$ which is the best L^1 approximation to f on the interval $(-1, 1)$ is the one that interpolates f on the points $\cos(k\pi/(d+2))$, $k = 1, 2, \ldots, d+1$ [3, p. 115]. So the best linear approximation to a convex function on $(-1, 1)$ interpolates on $\pm\frac{1}{2}$. Now the polynomial p_Q defined above does not interpolate f on $a \pm h/2$ (f is not a point function anyway), but it interpolates the averages of f. Precisely,

$$p_Q(a+h/2) = h^{-1}\int_a^{a+h} f \quad \text{and} \quad p_Q(a-h/2) = h^{-1}\int_{a-h}^{a} f.$$

Remark 2. The referee has kindly brought to our attention a relevant paper of S. Campanato [1]. If Q is a cube in $\mathbf{R}^n$, q (≥ 1) and λ (≥ 0) are real numbers, and k (≥ 0) is an integer, Campanato defines $\mathcal{L}_k^{(q,\lambda)}(Q)$ to be the set of functions $f \in L^q(Q)$ such that

$$\sup_{x_0 \in Q;\ 0>r\le d(Q)} \left(r^{-\lambda} \inf_{P \in \mathcal{P}_k} \int_{Q(x_0,r)} |f(x) - P(x)|^q\,dx\right)^{1/q} < \infty$$

where $\mathcal{P}_k$ is the set of all polynomials of degree $\le k$, $Q(x_0, r) = \{x \in Q:$

$|x - x_0| \leq r\}$, and $d(Q)$ is the diameter of Q. Note that $\mathfrak{L}_0^{(1,n)}(Q)$ is the set of functions of bounded mean oscillation on Q. What we have shown in this paper is $\mathfrak{L}_0^{(1,n)}(Q) = \mathfrak{L}_k^{(1,n)}(Q)$ for all integers n, $k \geq 1$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: General Electric Company, 450 Persian Drive, Sunnyvale, California 94086