

CONVEX COMBINATIONS OF UNIFORMLY MEAN STABLE MARKOV OPERATORS

ROBERT SINE¹

ABSTRACT. A convex combination of commuting uniformly mean stable Markov operators acting on $C(X)$ is shown to be uniformly mean stable. The proof is completely geometric.

Let $C(X)$ denote the B -space of continuous functions on a compact Hausdorff space. A (stable) Markov operator T is a positive ($Tf \geq 0$ whenever $f \geq 0$) linear transformation with $T1 = 1$. It is immediate that if S_i are Markov and $a_i \geq 0$ with $\sum a_i = 1$ for $1 \leq i \leq n$ then $T = \sum a_i S_i$ is again a Markov operator. We call any Markov operator T uniformly mean stable (u.m.s.) if the Cesàro means

$$A_n(T)f = (1/n)(I + T + \cdots + T^n)f$$

converge uniformly for every f in $C(X)$ (to a then necessarily continuous and invariant function). This strong convergence property is of interest in itself; moreover there is also a rather a complete ergodic decomposition for such operators (see [1], [2], [4], and [5]). Thus conditions which imply u.m.s. are of interest. For conditions which imply u.m.s. in the special cases of random point maps and random isometries, see [1] and the references given there. A more general classical result is that if T is uniquely ergodic (that is T has exactly one invariant probability) then T is u.m.s. (This may be found in Oxtoby's well-known review [3] for point maps.) This result, in turn, suggested the following [5, Theorem 2.2, but read \neq for the misprint $=$ there] which we will need here: T is u.m.s. if and only if the invariant functions of T separate the invariant probabilities of T . (This has been generalized to contractions on arbitrary B -spaces as well [6].) We will prove the

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Theorem. *The convex combination of commuting u.m.s. Markov operators is u.m.s.*

We make the observation here that it is sufficient to consider the convex combination of two operators. Thus we will take

$$T = \alpha R + \beta S$$

where α and β are nonnegative and sum to 1 while R and S are commuting Markov operators. The set of invariant probabilities of T will be denoted by $K(T)$ and similarly for R and S . Observe that these sets are each non-empty by almost anyone's fixed point theorem and, moreover, $K(R) \cap K(S) \neq \emptyset$ by almost everyone's (Riesz-Kakutani-Markov) fixed point theorem.

Lemma. *Let T be a convex combination of two commuting u.m.s. Markov operators. Then T is u.m.s. if and only if $K(T) = K(R) \cap K(S)$.*

Proof. We clearly have $K(T) \supset K(R) \cap K(S)$ in every case. Suppose first we have $K(T) = K(R) \cap K(S)$. Let μ_1 and μ_2 be two distinct T -invariant probabilities. Then since R is u.m.s and μ_1 and μ_2 are both in $K(R)$ there is a continuous R -invariant function f with $(f, \mu_1) \leq c < d \leq (f, \mu_2)$. Since S is u.m.s. we obtain a continuous limit by $g = \lim A_n(S)f$ where g is both R and S invariant. But then g is T -invariant and $(g, \mu_1) \leq c < d \leq (g, \mu_2)$. Since the T -invariant functions separate the T -invariant probabilities, T is u.m.s.

Now assume T is u.m.s. and let μ be a T -invariant probability. If μ is not in $K(R) \cap K(S)$ we use the geometric Hahn-Banach theorem to obtain a continuous function f so that for λ in $K(R) \cap K(S)$ we have $(f, \lambda) \leq c < d = (f, \mu)$. Let $g = \lim A_n(T)f$ and the same inequality will then hold for the T -invariant function g . Since $\int g d\mu = d$ the maximum of g on X is at least d . Let M be the maximizing set of g . That is $M = \{x: g(x) = \sup g\}$.

We call any closed set D with the property $T^*\delta(x)$ is supported in D for all x in D an invariant set (= self-supporting in [5]). Then M is a T -invariant set. But any T -invariant set is both R and S invariant as well. It follows at once that M supports a probability, say σ , that is both R and S invariant. Now $(g, \sigma) \leq c < d \leq (g, \sigma)$ and this contradiction shows $K(T) \subset K(R) \cap K(S)$ and thus finishes the proof.

Before using the Lemma to establish the Theorem we will make a comment using the Lemma in reverse. If R and S are commuting Markov matrices and $T = \alpha R + \beta S$ then $K(T) = K(R) \cap K(S)$. This is because of the well-known fact that all Markov matrices are u.m.s.

Proof of the Theorem. Let λ be an extreme point of the compact convex set $K(T)$. We observe that $T^*R^*\lambda = R^*T^*\lambda = R^*\lambda$ so that $R^*\lambda$ (and similarly $S^*\lambda$) is also a T -invariant probability. But $\alpha R^*\lambda + \beta S^*\lambda = T^*\lambda = \lambda$. So by extremality of λ we have $R^*\lambda = S^*\lambda = \lambda$. Thus the extremes of $K(T)$ are all in $K(R) \cap K(S)$. The Kreĭn-Mil'man theorem shows $K(T) \subset K(R) \cap K(S)$ and now the Lemma finishes the proof.

We base a second observation on the Lemma (together with the Theorem). If R and S commute and each is uniquely ergodic so $K(R) = K(S) = \{\mu\}$ then $T = \alpha R + \beta S$ is uniquely ergodic and $K(T) = K(R) = K(S) = \{\mu\}$.

It is not possible to derive u.m.s. in the reverse direction. That is T and R may be u.m.s. while S fails to be u.m.s. For let X be the Stone-Čech compactification of Z_+ . Let S be the extension to X of the shift on Z_+ . Let μ be a shift invariant probability on X and define R to be the projection $Rf = (f, \mu)$. Then $T = \frac{1}{2}(R + S)$ is not only u.m.s. for we even have T^n convergent in the strong operator topology (to R). Now R itself is u.m.s. but S has many invariant probabilities while only constants for invariant functions so cannot be u.m.s.

Finally we may ask what can be said if α and β are functions on X instead of constants. Under most circumstances αR and βS no longer commute and the instances in which they do are too specialized to be of interest. We will look at one example where the coefficient functions are allowed to vary over X . The convex combination will fail to be u.m.s. because of stagnation caused by the coefficients. Let X be the interval $[0, 1]$ and define

$$Tf(x) = xf(x) + (1 - x)f(x/2).$$

Note that $Sf = f(x/2)$ while R is the identity transformation. Thus $K(R)$ is the set of all probabilities on X while $K(S) = \{\delta(0)\}$. It is easy to show that $\delta(1)$ is T -invariant as well as $\delta(0)$. If T were u.m.s. there would be a continuous T -invariant function f with $f(0) = 0$ and $f(1) = 1$. But then there would be a point x_0 in X with $f(x_0/2) < f(x_0)$.

$$\begin{aligned} f(x_0) &= Tf(x_0) = x_0f(x_0) + (1 - x_0)f(x_0/2) \\ &< x_0f(x_0) + (1 - x_0)f(x_0) = f(x_0). \end{aligned}$$

This contradiction shows T is not u.m.s.

Added in proof. The key lemma used here is an inadvertent rediscovery of a result of M. Falkowitz [7]. The example shows that Falkowitz's result is

sharp in that the result may fail if the convex coefficients are continuous functions rather than constants.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT ALBANY,
ALBANY, NEW YORK 12222

Current address: Department of Mathematics, University of Rhode Island,
Kingston, Rhode Island 02881