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HIGHER DERIVATIONS ON FINITELY GENERATED INTEGRAL DOMAINS. II

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ABSTRACT. We prove

Theorem. *Let* $A = k[x_1, \ldots, x_m]$ *be a finitely generated integral domain over a field* k *of characteristic zero. Then* A *regular, i.e. the local ring* A_q *is regular for every prime ideal* $q \subsetneq A$, *is equivalent to the following two conditions*: (1) *no prime of* A *of height greater than one is differential, and* (2) *for all* $\phi \in \mathrm{Hom}_k(A, A)$, $\phi \in \mathrm{Der}_k^n(A)$ *if and only if* $\Delta\phi \in \sum_{i=1}^{n-1} \mathrm{Der}_k^i(A) \cup \mathrm{Der}_k^{n-i}(A)$ $(n = 1, 2, \ldots)$.

Here Δ denotes the Hochschild coboundary operator, $\cup$ denotes the cup product, and $\mathrm{Der}_k^n(R)$ is the module of higher derivations of rank n.

Introduction. Throughout this paper, $A = k[x_1, \ldots, x_m]$ will denote a finitely generated integral domain over a field k of characteristic zero. For each $n = 1, 2, \ldots,$ we shall let $\mathrm{Der}_k^n(A)$ denote the A-module of all nth order derivations of A to A which vanish on k. Thus, $\phi \in \mathrm{Der}_k^n(A)$ if and only if $\phi \in \mathrm{Hom}_k(A, A)$, and for all $a_0, \ldots, a_n \in A$ we have

$$\phi(a_0 a_1 \cdots a_n) = \sum_{s=1}^{n} (-1)^{s-1} \sum_{i_1 \cdots i_s} a_{i_1} \cdots a_{i_s} \phi(a_0 \cdots \check{a}_{i_1} \cdots \check{a}_{i_s} \cdots a_n). \tag{1}$$

The author refers the reader to [5] for the various properties of the modules $\mathrm{Der}_k^n(A)$ which are used in this paper.

It follows from [5, Proposition 4 and Corollary 6.1] that any composite $\delta_1 \ldots \delta_j$ $(1 \leq j \leq n)$ of j-derivations $\delta_i \in \mathrm{Der}_k^1(A)$ is an nth order derivation. The A-submodule of $\mathrm{Der}_k^n(A)$ spanned by all such composites will be denoted by $\mathrm{der}_k^n(A)$.

We shall say that a prime ideal p in A is differential if $\delta(p) \subseteq p$ for all $\delta \in \mathrm{Der}_k^1(A)$. We shall call A a regular ring if for all prime ideals $q \subsetneq A$,

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the local ring A_q is regular. With these definitions, the author proved [1]

Theorem 1. *Let $A = k[x_1, \ldots, x_m]$ be a finitely generated integral domain over a field k of characteristic zero. Then A is a regular ring if and only if the following two properties are satisfied*: (a) *no prime p of A of height greater than one is differential and* (b) $\mathrm{Der}_k^n(A) = \mathrm{der}_k^n(A)$ *for all n.*

Thus, the arithmetic condition (b) on the modules $\mathrm{Der}_k^n(A)$ in the presence of (a) implies regularity. The purpose of this paper is to show that the condition (b) can be replaced by another quite different arithmetic condition on the modules $\mathrm{Der}_k^n(A)$.

Recall that if ϕ and ψ are k-linear mappings of A into itself, then the Hochschild coboundary $\Delta\phi$ of ϕ is the k-bilinear mapping $\Delta\phi\colon A \times A \longrightarrow A$ given by $\Delta\phi(a, b) = \phi(ab) - a\phi(b) - b\phi(a)$ for all $a, b \in A$. The cup product $\phi \cup \psi\colon A \times A \longrightarrow A$ is the k-bilinear mapping defined by $\phi \cup \psi(a, b) = \phi(a)\psi(b)$. If P and Q are two A-submodules of $\mathrm{Hom}_k(A, A)$, then $P \cup Q$ will denote the set of all k-bilinear mappings of $A \times A \longrightarrow A$ which are finite sums of mappings of the form $\phi \cup \psi$ for $\phi \in P$, $\psi \in Q$.

Let us set

$$\sum(A) = \sum_{i=1}^{n-1} \mathrm{Der}_k^i(A) \cup \mathrm{Der}_k^{n-i}(A).$$

Then it follows from [5, Proposition 3] that $\phi \in \mathrm{Der}_k^n(A)$ if $\Delta\phi \in \Sigma(A)$. In [3], Y. Ishibashi proved

Theorem 2. *Let A be a finitely generated algebra over a field k such that $\Omega_k^i(A)$ is A-projective for every $i \geq 1$. Let $\phi \in \mathrm{Hom}_k(A, A)$. Then $\phi \in \mathrm{Der}_k^n(A)$ if and only if $\Delta\phi \in \Sigma(A)$. Here $\Omega_k^i(A)$ denotes the A-module of ith order differentials.*

If A is a finitely generated integral domain over a field k of characteristic zero, then we shall prove a partial converse to Theorem 2.

Main results.

Theorem 3. *Let $A = k[x_1, \ldots, x_m]$ be a finitely generated integral domain over a field k of characteristic zero. Then A is a regular ring if and only if the following two conditions are satisfied*:

(a) *no prime ideal p of A of height greater than one is differential; and*

(b) *for every positive integer n and every $\phi \in \mathrm{Hom}_k(A, A)$ $\phi \in \mathrm{Der}_k^n(A)$ if and only if $\Delta\phi \in \Sigma(A)$.*

Proof. Suppose A is a regular ring. Then by Theorem 1, A satisfies

(a) and for each positive integer n, $\mathrm{Der}_k^n(A) = \mathrm{der}_k^n(A)$. But if every $\phi \in \mathrm{Der}_k^n(A)$ is a linear combination of composites of first order derivations, then a simple induction argument shows $\Delta\phi \in \Sigma(A)$. Thus, one direction of Theorem 3 follows immediately from Theorem 1.

Now suppose A satisfies conditions (a) and (b). We shall first show that A is integrally closed. Let Q denote the quotient field of A, and let $\overline{A}$ denote the integral closure of A in Q. Let us assume that $A \neq \overline{A}$. Then the conductor C of A is a proper ideal in A. By [6, Corollary, p. 169], C is a differential ideal. Let p be an associated prime of C. Then by [7, Theorem 1], p is differential, and thus by condition (a) p is a minimal prime of A.

Now let $R = A_p$ (A localized at p). Then $\overline{R}$, the integral closure of R in Q, is given by $\overline{A}_p$. The conductor of R in $\overline{R}$ is just CR. We note that $\overline{R}$ is a semilocal ring with maximal ideals say $p_1, \ldots, p_s$. If we set $V_i = \overline{R}_{p_i}$, $i = 1, \ldots, s$, then V_i is a discrete rank one valuation ring, and $\overline{R} = \bigcap_{i=1}^s V_i$. Since $\Omega_k^n(R) = \Omega_k^n(A) \otimes_A A_p$ [5, Theorem 9], and A_p is a flat A-module, we see that $\mathrm{Der}_k^n(R) = \mathrm{Der}_k^n(A) \otimes_A A_p$. So any derivation ϕ in $\mathrm{Der}_k^n(R)$ has the form $(1/s)\phi'$ with $\phi' \in \mathrm{Der}_k^n(A)$, s an element of A not in p. Thus, since A satisfies (b), one easily sees that if $\phi \in \mathrm{Der}_k^n(R)$, then $\Delta\phi \in \Sigma(R)$. On the other hand, if $\phi \in \mathrm{Hom}_k(R, R)$, and $\Delta\phi \in \Sigma(R)$, then it follows from [5, Proposition 3] that $\phi \in \mathrm{Der}_k^n(R)$. Thus, R satisfies condition (b) in the theorem.

Let the tanscendence degree of Q over k be r, and let $\Omega_k^1(T)$ denote the module of first order differentials of any k-algebra T. It follows from [4, Theorem 3′] that $\Omega_k^1(V_i)$ is a free V_i-module necessarily of rank r. Since $\overline{R}$ is semilocal, it follows that $\mathrm{Der}_k^1(\overline{R})$ is a free $\overline{R}$-module of rank r. Let $\overline{d}\colon \overline{R} \to \Omega_k^1(\overline{R})$ denote the canonical derivation of $\overline{R}$ into $\Omega_k^1(\overline{R})$.

Since the depth of p is $r - 1$, the quotient field of A/p has transcendence degree $r - 1$ over k. Hence there exist elements $\alpha_1, \ldots, \alpha_{r-1} \in A - p$ such that the quotient field of A/p is a separable algebraic extension of $k(\overline{\alpha}_1, \ldots, \overline{\alpha}_{r-1})$. Here $\overline{\alpha}_i$ of course denotes the image of α_i in A/p. Note that the field $F = k(\alpha_1, \ldots, \alpha_{r-1})$ is contained in R.

Using [8, Theorem 18, p. 45], we can find an element $\beta \in \bigcap_{i=1}^s p_i$ such that β generates the maximal ideal in each local ring V_i, $i = 1, \ldots, s$. It now follows from the proof of [4, Theorem 3′] and Nakayama's lemma that $\{\overline{d}(\beta), \overline{d}(\alpha_1), \ldots, \overline{d}(\alpha_{r-1})\}$ is a free $\overline{R}$-module basis of $\Omega_k^1(\overline{R})$. Since $\mathrm{Hom}_{\overline{R}}(\Omega_k^1(\overline{R}), \overline{R}) \cong \mathrm{Der}_k^1(\overline{R})$, there exist derivations $\delta_0, \delta_1, \ldots, \delta_{r-1} \in \mathrm{Der}_k^1(\overline{R})$ such that the following equations are satisfied:

(2) $$\delta_0(\beta) = 1, \quad \delta_0(\alpha_i) = 0 = \delta_i(\beta), \qquad i = 1, \ldots, r-1,$$

and

$$\delta_i(\alpha_j) = \begin{cases} 0 & \text{if } i \neq j,\ i, j = 1, \ldots, r-1, \\ 1 & \text{if } i = j. \end{cases}$$

Clearly $\{\delta_0, \ldots, \delta_{r-1}\}$ is a free $\overline{R}$-module basis of $\mathrm{Der}_k^1(\overline{R})$ and also a Q-algebra basis for $\bigcup_{n=1}^{\infty} \mathrm{Der}_k^n(Q)$.

If $\phi \in \mathrm{Der}_k^n(R)$, then by [5, Theorem 15] $\phi \in \mathrm{Der}_k^n(Q)$. Thus, ϕ can be written uniquely as a polynomial (with coefficients in Q) in $\delta_0, \ldots, \delta_{r-1}$ of degree less than or equal to n. Let $J = \bigcap_{i=1}^{s} p_i$, the Jacobson radical of $\overline{R}$. We need the following

Lemma. *Let $\phi \in \mathrm{Der}_k^n(R)$. Then $\phi = g(\delta_0, \ldots, \delta_{r-1})$, where $g(X_0, \ldots, X_{r-1})$ is a polynomial in indeterminates X_i with coefficients in Q. Furthermore, the coefficient of δ_0^n in g is an element of J^n.*

Proof. Only the last statement in the Lemma needs proof. We proceed via induction on n. Suppose $\phi \in \mathrm{Der}_k^1(R)$. By [6, Corollary, p. 169], CR is differential under ϕ. Since $\phi \in \mathrm{Der}_k^1(\overline{R})$, $C\overline{R}$ is differential under ϕ. Thus, every associated prime of $C\overline{R}$ is differential under ϕ. Since every p_i is an associated prime of $C\overline{R}$, we have $\phi(p_i) \subseteq p_i$, $i = 1, \ldots, s$. Thus, $\phi(\beta) \in J$. But if $\phi = \sum_{i=0}^{r-1} a_i\delta_i$, $a_i \in Q$, then $\phi(\beta) = a_0$. Thus the Lemma is proven in the case $n = 1$.

We shall sketch the proof of the $n = 2$ case before proceeding to the inductive step. Suppose $\phi \in \mathrm{Der}_k^2(R)$. Since R satisfies condition (b), there exist $\lambda_l, \psi_l \in \mathrm{Der}_k^1(R)$, and elements $e_l \in R$ such that for all a and b in R we have

(3) $$\phi(ab) = a\phi(b) + b\phi(a) + \sum_l e_l\psi_l(a)\lambda_l(b).$$

One easily checks using [5, Theorem 15] that (3) holds for all $a, b \in Q$. Now write

(4) $$\phi = \sum_{i=0}^{r-1} a_i\delta_i + \sum_{i_1, i_2=0}^{r-1} a_{i_1 i_2}\delta_{i_1}\delta_{i_2} \quad \text{with } a_i, a_{i_1 i_2} \in Q$$

and

(5) $$\psi_l = \sum_{i=0}^{r-1} \mu_{il}\delta_i, \quad \lambda_l = \sum_{i=0}^{r-1} \gamma_{il}\delta_i \quad \text{with } \mu_{il}, \gamma_{il} \in Q.$$

From the case $n = 1$, we have $\mu_{0l}, \gamma_{0l} \in J$. Now substitute (4) and (5) into (3) and obtain

$$\sum_{i_1,i_2} a_{i_1i_2}(\delta_{i_1}(a)\delta_{i_2}(b) + \delta_{i_1}(b)\delta_{i_2}(a)) \tag{6}$$
$$= \sum_l e_l\left(\sum_{i=0}^{r-1} \mu_{il}\delta_i(a)\right)\left(\sum_{i=0}^{r-1} \gamma_{il}\delta_i(b)\right).$$

If we now substitute $a = b = \beta$ in (6), we get $2a_{00} = \Sigma_l e_l\mu_{0l}\gamma_{01} \in J^2$. Thus $a_{00} \in J^2$ and the case $n = 2$ is complete.

Assume we have now proven the Lemma for $t = 1, 2, \ldots, n-1$ $(n \geq 3)$, and consider $\phi \in \mathrm{Der}_k^n(R)$. Then by condition (b) there exist elements $e_{l,j} \in R$ and derivations $\psi_l^{(j)}, \lambda_l^{(j)} \in \mathrm{Der}_k^j(R)$, $j = 1, \ldots, n-1$, such that for all $a, b \in R$ we have

$$\phi(ab) = a\phi(b) + b\phi(a) + \sum e_{l1}\psi_l^{(1)}(a)\lambda_l^{(n-1)}(b) \tag{7}$$
$$+ \sum_l e_{l2}\psi_l^{(2)}(a)\lambda_l^{(n-2)}(b) + \cdots + \sum_l e_{ln-1}\psi_l^{(n-1)}(a)\lambda_l^{(1)}(b).$$

Now write ϕ as a polynomial $g(\delta_0, \ldots, \delta_{r-1})$ of degree less than or equal to n. If the coefficient of δ_0^n in g is zero, then we have nothing to prove. Thus, without loss of generality, we can assume that the coefficient of δ_0^n in g is not zero. By the induction hypothesis, we can write each $\psi_l^{(j)}$ and $\lambda_l^{(j)}$ as a polynomial in $\delta_0, \delta_1, \ldots, \delta_{r-1}$ whose coefficient of δ_0^j is in J^j. Say

$$\psi_l^{(j)} = \sum c_t^{l,j}\delta_t + \sum c_{t_1t_2}^{l,j}\delta_{t_1}\delta_{t_2} + \cdots + \sum c_{t_1\cdots t_j}^{l,j}\delta_{t_1}\cdots\delta_{t_j} \tag{8}$$

and

$$\lambda_l^{(j)} = \sum b_t^{l,j}\delta_t + \sum b_{t_1t_2}^{l,j}\delta_{t_1}\delta_{t_2} + \cdots + \sum b_{t_1\cdots t_j}^{l,j}\delta_{t_1}\cdots\delta_{t_j}.$$

We now substitute $\phi = g(\delta_0, \ldots, \delta_{r-1})$ and the expressions in (8) into (7). Using the relations in (2), we can eliminate all terms in (7) which involve any δ_i, $i = 1, \ldots, r-1$. Thus, (7) reduces to an equation of the form

$$\sum_{i=1}^n a_i\delta_0^i(ab) = a\left(\sum_{i=1}^n a_i\delta_0^i(b)\right) + b\left(\sum_{i=1}^n a_i\delta_0^i(a)\right) \tag{9}$$
$$+ \sum_l e_{l1}(c_0^{l,1}\delta_0(a))(b_0^{l,n-1}\delta_0 + \cdots + b_{0\cdots0}^{l,n-1}\delta_0^{n-1})(b)$$
$$+ \cdots + \sum_l e_{l,n-1}(c_0^{l,n-1}\delta_0 + \cdots + c_{0\cdots0}^{l,n-1}\delta_0^{n-1})(a)(b_0^{l,1}\delta_0(b)).$$

Here $c^{l,j}_{0\dots 0}, b^{l,j}_{0\dots 0} \in J^j$ for each $j = 1, \dots, n-1$, when the number of zeros appearing in the subscript is j.

If we now make various substitutions of the form $a = \beta^s$, $b = \beta^t$ and eliminate sums which are equal, we easily see that $a_n \in J^n$. Since a_n is the coefficient of δ_0^n appearing in g, the proof is complete.

Using the Lemma, we can easily argue that A is integrally closed. Let $c \in CR$. Then $c\delta_0^n \in \mathrm{Der}_k^n(R)$ for every n. Thus by the Lemma, $c \in J^n$. Since c is arbitrary, we have $CR \subset \bigcap_{n=1}^{\infty} J^n$. Since J is the Jacobson radical of $\overline{R}$, we have $\bigcap_{n=1}^{\infty} J^n = 0$. Thus $CR = (0)$. But we are assuming $A \neq \overline{A}$, and thus $CR \neq (0)$, a contradiction. Thus, we have shown that conditions (a) and (b) imply that A is integrally closed.

We now show that A is a regular ring. Let q be a minimal prime of A. Then A_q is a discrete rank one valuation ring. In particular, A_q is a regular local ring. Assume we have shown that A_q is a regular local ring for all primes q of height less than or equal to t. Let q be a prime of height $t+1$. Then by (a) q is not differential under $\mathrm{Der}_k^1(A)$. Then qA_q is not differential under $\mathrm{Der}_k^1(A_q)$. We note that the induction hypothesis implies that every proper localization of A_q is regular. It now follows from [7, Theorem 5] that A_q is regular. This completes the proof of Theorem 3.

It follows from [2, 16.12.12] that if A is regular, then A is a smooth algebra over k. Thus Theorem 3 can be viewed as a partial converse of Theorem 2.

Example. In studying the proofs of Theorems 1 and 3, we see that the following theorem proven by A. Seidenberg [6] is of crucial importance.

Theorem 4. *Let R be an integral domain containing the rational numbers k, and let R' be the ring of elements in the quotient field of R which are quasi-integral over R. Let $\delta \in \mathrm{Der}_k^1(R)$. Then $\delta \in \mathrm{Der}_k^1(R')$.*

It is well known that this theorem is false if k is a field of characteristic not zero. It is somewhat surprising that this theorem is also false (in the characteristic zero case) for higher derivations. We give an example.

Consider the curve $x^2 = y^3$ over the rational numbers k. That is, let $A = k[X, Y]/(X^2 - Y^3) = k[x, y]$. One easily checks that A is an integral domain whose integral closure $\overline{A}$ in the quotient field Q of A is given by $\overline{A} = A[x/y]$. Since Q is a separable algebraic extension of $k(y)$, it follows that any $\phi \in \mathrm{Der}_k^2(Q)$ is uniquely determined by its values on y and y^2. A simple calculation shows that if $\phi(y) = a$ and $\phi(y^2) = b$, then

$$\phi(x) = \frac{3y}{8}\left(\frac{2ya + b}{x}\right), \quad \phi(x^2) = 3yb - 3y^2a \quad \text{and} \quad \phi(xy) = \frac{5y^2}{8}\left(\frac{3b - 2ya}{x}\right).$$

In particular, if we set $a = 1$ and $b = -2y$, then $\phi \in \mathrm{Der}_k^2(A)$ and is given by

(10) $\phi(y) = 1, \quad \phi(y^2) = -2y, \quad \phi(x) = 0, \quad \phi(x^2) = -9y^2 \quad \text{and} \quad \phi(xy) = -5x.$

The conductor C of A is given by $C = (x, y)$. Note that $\phi(C) \not\subset C$. Using [5, Theorem 15], we see that $\phi(x/y) = x/y^2 \notin \overline{A}$. Thus, higher derivations on A need not map $\overline{A}$ into $\overline{A}$.

Finally the author notes that in the above example (and all other examples investigated so far) if $\phi \in \mathrm{Der}_k^2(A)$ satisfies $\Delta\phi \in \mathrm{Der}_k^1(A) \cup \mathrm{Der}_k^1(A)$, then $\phi(\overline{A}) \subset \overline{A}$. This leads to the following conjecture. Suppose A is a finitely generated integral domain over a field k of characteristic zero. Let $\phi \in \mathrm{Der}_k^n(A)$, and suppose $\Delta\phi \in \Sigma(A)$. Suppose further that the derivations $\psi_l^{(i)}$, $\lambda_l^{(i)}$ (as in (7)) used in $\Delta\phi$ map the integral closure $\overline{A}$ of A into itself. Then ϕ maps $\overline{A}$ to $\overline{A}$.

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