APPLICATIONS OF GRAPH THEORY TO MATRIX THEORY

FRANK W. OWENS

ABSTRACT. Let $A_1, \ldots, A_k$ be $n \times n$ matrices over a commutative ring $R$ with identity. Graph theoretic methods are established to compute the standard polynomial $[A_1, \ldots, A_k]$. It is proved that if $k < 2n - 2$, and if the characteristic of $R$ either is zero or does not divide $4l(n/2) - 2$, where $l$ denotes the greatest integer function, then there exist $n \times n$ skew-symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$.

1. Introduction. Let $A_1, \ldots, A_k$ be $n \times n$ matrices over a commutative ring $R$ with identity. Let $S_k$ be the symmetric group of degree $k$. Define the standard polynomial $[A_1, \ldots, A_k]$ by

$$[A_1, \ldots, A_k] = \sum \text{sgn} \sigma A_{\sigma(1)} \cdots A_{\sigma(k)},$$

where the summation is over all permutations $\sigma \in S_k$.

Amitsur and Levitzki proved algebraically [1] that $[A_1, \ldots, A_k] = 0$ if $k \geq 2n$. Their proof is elementary but lengthy. Swan gave a simpler and shorter graph theoretic proof of their theorem [8], [9]. Amitsur and Levitzki also proved that if $k < 2n$, then there exist $n \times n$ matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$, i.e., their theorem is sharp. See [1], [7] and [8] for examples. It is known [7] that if $k < 2n$, then there exist $n \times n$ symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$.

Kostant proved in [3] that $[A_1, \ldots, A_k] = 0$ if $k \geq 2n - 2$, where $n$ is even, and each of the matrices $A_j$ is complex skew-symmetric. In this paper we prove using graph theoretic methods that if $k < 2n - 2$, and if the characteristic of $R$ either equals 0 or does not divide $4l(n/2) - 2$, where $l$ denotes the greatest integer function, then there exist $n \times n$ skew-symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$. This solves Conjecture 2 in [7] in the affirmative. In particular, for $n$ even this implies that Kostant's theorem is sharp. Kostant's proof is nonelementary and uses cohomology theory. In [4] we present a graph theoretic proof that $[A_1, \ldots, A_k] = 0$ for $k \geq 2n - 2$.

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if each of the matrices $A_j$ is skew-symmetric, and $R$ is an integral domain not of characteristic 2. This generalization of Kostant’s theorem solves Conjecture 1 in [7] in the affirmative. The solutions of Conjectures 1 and 2 have been obtained independently by Hutchinson [2] and by Rowen [6]. Our results in this paper are somewhat stronger for the case when the characteristic of $R$ is larger than 2. See [2], [5] and [6] for related results.

2. Algebraic preliminaries. We first state some algebraic properties of $[\ldots, \ldots]$. $[\ldots, \ldots]$ is alternating and multilinear. $[\ldots, \ldots]$ may be defined recursively by $[A] = A$ and

$$
[A_1, \ldots, A_k] = \sum_{j=1}^{k} (-1)^{j-1} [A_1, \ldots, \hat{A}_j, \ldots, A_k]\]
$$

$$
= \sum_{j=1}^{k} (-1)^{k-j} [A_1, \ldots, \hat{A}_j, \ldots, A_k]A_j \quad \text{for } k > 1,
$$

where the notation $\hat{A}_j$ means that the matrix $A_j$ is absent.

The following propositions are easily established using the alternating and multilinear properties of $[\ldots, \ldots]$. Let $A'$ denote the transpose of the matrix $A$.

**Proposition 1.** If $A_1, \ldots, A_k$ are $n \times n$ matrices, then

(a) $[A_1, \ldots, A_k]' = [A_1', \ldots, A_k']$.

(b) $[A_1, \ldots, A_k] = (-1)^{k(k-1)/2} [A_1', \ldots, A_k]$. 

(c) $[A_1, \ldots, A_k]' = (-1)^{k(k-1)/2} [A_1', \ldots, A_k']$.

**Proposition 2.** If each $A_j$ is an $n \times n$ skew-symmetric matrix, then

(a) $[A_1, \ldots, A_k]' = (-1)^{k(k+1)/2} [A_1, \ldots, A_k]$.

(b) $[A_1, \ldots, A_k]$ is symmetric iff $k \equiv 0$ or $3$ (mod 4) and is skew-symmetric iff $k \equiv 1$ or $2$ (mod 4).

**Proposition 3.** If each $A_j$ is an $n \times n$ symmetric matrix, then

(a) $[A_1, \ldots, A_k]' = (-1)^{k(k-1)/2} [A_1, \ldots, A_k]$.

(b) $[A_1, \ldots, A_k]$ is symmetric iff $k \equiv 0$ or $1$ (mod 4) and is skew-symmetric iff $k \equiv 2$ or $3$ (mod 4).

Let $e_{ij}$ be the elementary matrix unit which has a 1 in the $(i, j)$th position and zeros elsewhere. Let $s_{ij}$, $1 \leq i < j \leq n$, denote the $n \times n$ skew-symmetric matrix unit $e_{ij} - e_{ji}$. Let $t_{ij}$, $1 \leq i \leq j \leq n$, denote the $n \times n$ symmetric matrix unit $e_{ij} + (1 - \delta_{ij})e_{ji}$, where $\delta_{ij}$ is the Kronecker delta. Then
\begin{align}
(2.1) \quad e_{ij}e_{il} &= \delta_{ij}e_{il} \\
{s_{ij}^s h' &= (e_{ij} - e_{ii})(e_{il} - e_{il}) = e_{ij}e_{il} - e_{ij}e_{ib} - e_{ij}e_{hl} + e_{ij}e_{lh} \\
&= \delta_{ij}e_{il} - \delta_{ij}e_{ib} - \delta_{ih}e_{jl} + \delta_{il}e_{jh}.
\end{align}

\begin{align}
(2.2) \quad t_{ij}e_{il} &= (e_{ij} + (1 - \delta_{ij})e_{il})(e_{il} + (1 - \delta_{il})e_{hl}) \\
&= e_{ij}e_{il} + (1 - \delta_{il})e_{ij}e_{hl} + (1 - \delta_{ij})e_{il}e_{hl} + (1 - \delta_{ij})(1 - \delta_{il})e_{ij}e_{hl} \\
&= \delta_{ij}e_{il} + (1 - \delta_{il})\delta_{ij}e_{hl} + (1 - \delta_{ij})\delta_{ih}e_{jl} + (1 - \delta_{ij})(1 - \delta_{ih})\delta_{il}e_{jh}
\end{align}

\begin{align}
(2.3) \quad \begin{cases}
0 & \text{if } j < b, \\
\delta_{il}e_{il} & \text{if } j = b, \\
\delta_{ih}e_{il} & \text{if } b < j < l, \\
\delta_{ij}e_{il} + (1 - \delta_{ih})e_{ib} + (1 - \delta_{ij})\delta_{ih}e_{jl} & \text{if } j = l, \\
\delta_{ij}e_{il} + (1 - \delta_{ih})\delta_{il}e_{jh} & \text{if } j > l.
\end{cases}
\end{align}

3. Graph theoretic preliminaries. Let $G$ be a graph having $n$ vertices $v_1, \ldots, v_n$ and $k$ edges $e_1, \ldots, e_k$. If $v_i$ and $v_j$ are vertices of $G$, then an Euler path in $G$ from $v_i$ to $v_j$ is a permutation $\omega \in S_k$ for which there exists an orientation of $G$ such that

(a) $e_{\omega_1}$ starts at $v_i$, i.e., $v_i$ is the initial vertex of $e_{\omega_1}$,
(b) $e_{\omega_k}$ ends at $v_j$, i.e., $v_j$ is the terminal vertex of $e_{\omega_k}$, and
(c) the terminal vertex of $e_{\omega_h}$ is the initial vertex of $e_{\omega(h+1)}$ for $1 \leq h < k$.

If, in addition, $G$ is a digraph, then a unicursal path \cite{8} in $G$ from $v_i$ to $v_j$ is an Euler path in $G$ from $v_i$ to $v_j$ with respect to the given orientation of $G$. Thus if $G$ is a digraph, every unicursal path in $G$ from $v_i$ to $v_j$ is also an Euler path in $G$ from $v_i$ to $v_j$ but not conversely, i.e. we deal only with the given directions of the edges when considering unicursal paths.

If $G$ is a digraph without loops and $\omega$ is an Euler path in $G$ from $v_i$ to $v_j$, then some of the edges of $G$ may have directions induced by $\omega$ opposite to their given directions. We refer to the number of such edges by $r(\omega)$. Thus
the number of edges of $G$ which have directions induced by $\omega$ which are the same as their given directions is $k - \tau(\omega)$.

Let each $A_j$ be some $e_{hl}$. Define a digraph $G$ as follows. $G$ has $n$ vertices $v_1, \ldots, v_n$, and $G$ has a directed edge $e_j$ from $v_h$ to $v_l$ for each $A_j = e_{hl}$. The following theorem is due to Swan [8] and is immediate from multiplication rule (2.1).

**Theorem 1.** The $(i, j)$th entry in $[A_1, \ldots, A_k]$ is $\Sigma \text{sgn}(\omega)$, where the summation is over all unicursal paths $\omega$ in $G$ from $v_i$ to $v_j$.

Let each $A_j$ be some $s_{hl}$. Define a digraph $G$ as follows. $G$ has $n$ vertices $v_1, \ldots, v_n$, and $G$ has a directed edge $e_j$ from $v_h$ to $v_l$ for each $A_j = s_{hl}$. The next theorem follows immediately from Theorem 1 and multiplication rule (2.2).

**Theorem 2.** The $(i, j)$th entry in $[A_1, \ldots, A_k]$ is $\Sigma (-1)^{\tau(\omega)} \text{sgn}(\omega)$, where the summation is over all Euler paths $\omega$ in $G$ from $v_i$ to $v_j$.

Let each $A_j$ be some $t_{hl}$. Define a graph $G$ as follows. $G$ has $n$ vertices $v_1, \ldots, v_n$, and $G$ has an edge $e_j$ from $v_h$ to $v_l$ for each $A_j = t_{hl}$. The next theorem follows immediately from Theorem 1 and multiplication rule (2.3).

**Theorem 3.** The $(i, j)$th entry in $[A_1, \ldots, A_k]$ is $\Sigma \text{sgn}(\omega)$, where the summation is over all Euler paths $\omega$ in $G$ from $v_i$ to $v_j$.

It is easy to give a similar graph theoretic interpretation to $[A_1, \ldots, A_k]$ when the $A_j$'s are a mixture of elementary, skew-symmetric and symmetric matrix units.

4. Main result. This section establishes

**Theorem 4.** If $k < 2n - 2$, and if the characteristic of $R$ either equals 0 or does not divide $4l(\lfloor n/2 \rfloor) - 2$, where 1 denotes the greatest integer function, then there exist $n \times n$ skew-symmetric matrices $A_1, \ldots, A_k$ such that $[A_1, \ldots, A_k] \neq 0$.

By the recursion formula for $[A_1, \ldots, A_k]$ in §2 it is sufficient to prove Theorem 4 for the case $k = 2n - 3$, $n > 1$. For this case let the matrices $A_1, \ldots, A_k$ be $s_{12}, s_{23}, s_{13}, s_{34}, s_{24}, \ldots, s_{n-1,n}, s_{n-2,n}$, and let $B_n = [A_1, \ldots, A_k]$. By direct computation $B_2 = s_{12}$ and $B_3 = -2(e_{11} + e_{22} + e_{33})$. Hence, Theorem 4 is true for $n = 2$ or 3. Figure 1 illustrates a portion of the digraph $G_n$ described preceding Theorem 2 associated with $B_n$. 
Lemma 1. If \( n > 3 \), then there exists an integer \( c_n \) such that \( B_n = c_n(e_{2,n-1} + (-1)^{n-1}e_{n-1,2}) \), i.e., \( B_n = c_n s_{2,n-1} \) for \( n \) even > 3, and \( B_n = c_n t_{2,n-1} \) for \( n \) odd > 3.

Proof. Proposition 2 implies that \( B_n \) is skew-symmetric iff \( n \) is even and is symmetric iff \( n \) is odd. \( v_2 \) and \( v_{n-1} \) are the only vertices of \( G_n \) of odd order since \( n > 3 \). Hence, the only possible Euler paths \( \omega \) in \( G_n \) are from \( v_2 \) to \( v_{n-1} \) or from \( v_{n-1} \) to \( v_2 \). Apply Theorem 2.

The first few computed \( c_n \)'s are \( c_4 = c_5 = -6 \), \( c_6 = c_7 = -10 \) and \( c_8 = -14 \). For later convenience we set \( c_2 = c_3 = -2 \).

This paragraph is not necessary for the proof of the theorem but may be of interest. If \( E_n \) denotes the number of Euler paths \( \omega \) in \( G_n \) from \( v_2 \) to \( v_{n-1} \), then \( E_2 = 1 \), \( E_3 = 2 \), \( E_4 = 6 \), \( E_5 = 16 \), \( E_6 = 44 \), \( E_7 = 120 \) and \( E_8 = 328 \). We set \( E_1 = 0 \). Then the number of such Euler paths

(a) with \( \omega_1 = 1 \) is \( E_{n-1} \) for \( n > 1 \),
(b) with \( \omega_1 = 2 \) is \( E_{n-1} \) for \( n > 1 \), and
(c) with \( \omega_1 = 5 \) is \( 2E_{n-2} \) for \( n > 2 \).

(a), (b) and (c) imply that the \( E_n \)'s satisfy the difference equation \( E_n = 2E_{n-1} + 2E_{n-2} \) for \( n > 2 \) with the initial conditions \( E_1 = 0 \) and \( E_2 = 1 \) whose solution is

\[
E_n = (\sqrt{3}/6)((1 + \sqrt{3})^{n-1} - (1 - \sqrt{3})^{n-1}).
\]

Lemma 2. Let \( P_n \) denote the set of all Euler paths in \( G_n \) from \( v_2 \) to \( v_{n-1} \). Then \( \Sigma (-1)^{r(\omega)} \text{sgn}(\omega) \) equals

(a) \( c_{n-1} \) for \( n > 3 \), where the summation is over all \( \omega \in P_n \) such that \( \omega_1 = 1 \),
(b) \( c_{n-2} - c_{n-3} \) for \( n > 4 \), where the summation is over all \( \omega \in P_n \) such that \( \omega_1 = 2 \),
(c) 0 for \( n > 4 \), where the summation is over all \( \omega \in P_n \) such that \( \omega_1 = 5 \).
Proof of (a). \( \omega_2 = 3 \) for each \( \omega \in P_n \) such that \( \omega_1 = 1 \). Define \( f_j \) by
\[
f_j(x) = x + j \text{ and } f: \{2, 4, 5, \ldots, 2n - 3\} \rightarrow \{1, 2, 3, \ldots, 2n - 5\} \text{ by } f(2) = 1 \text{ and } f(x) = x - 2 \text{ for } 4 \leq x \leq 2n - 3.
\]
Define \( F: \{\omega \in P_n | \omega_1 = 1\} \rightarrow P_{n-1} \) by \( \omega' = F(\omega) = f \circ \omega \circ f_2 \). \( F \) is a 1-1 correspondence such that \( r(\omega) = r(\omega') + 1 \) and \( \text{sgn}(\omega) = -\text{sgn}(\omega') \) for each \( \omega \in P_n \) such that \( \omega_1 = 1 \). Therefore, \( c_n-1 = \sum (-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega' \in P_{n-1} \).

Proof of (b). By direct computation \( \sum (-1)^{r(\omega')} \text{sgn}(\omega) = 0 = c_3 - c_2 \)
where the summation is over all \( \omega \in P_5 \) such that \( \omega_1 = 2 \). Hence, we may assume that \( n > 5 \). \( \omega_2 = 3, 4 \) or \( 7 \) for each \( \omega \in P_n \) such that \( \omega_1 = 2 \). \( \omega_3 = 1 \) and \( \omega_4 = 5 \) for each \( \omega \in P_n \) such that \( \omega_1 = 2 \) and \( \omega_2 = 3 \). Define \( f: \{4, 6, 7, \ldots, 2n - 3\} \rightarrow \{1, 2, 3, \ldots, 2n - 7\} \) by \( f(4) = 1 \) and \( f(x) = x - 4 \) for \( 6 \leq x \leq 2n - 3 \). Define \( F: \{\omega \in P_n | \omega_2 = 3\} \rightarrow P_{n-2} \) by \( \omega' = F(\omega) = f \circ \omega \circ f_4 \). \( F \) is a 1-1 correspondence such that \( r(\omega) = r(\omega') + 1 \) and \( \text{sgn}(\omega) = -\text{sgn}(\omega') \) for each \( \omega \in P_n \) such that \( \omega_1 = 2 \) and \( \omega_2 = 3 \). Therefore, \( c_{n-2} = \sum (-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega' \in P_{n-2} \).

\( \omega_3 = 5, 6 \) or \( 9 \) for each \( \omega \in P_n \) such that \( \omega_1 = 2 \) and \( \omega_2 = 4 \). \( \omega_4 = 1 \), \( \omega_5 = 3 \) and \( \omega_6 = 7 \) for each \( \omega \in P_n \) such that \( \omega_1 = 2, \omega_2 = 4 \) and \( \omega_3 = 5 \). Define \( f: \{6, 8, 9, \ldots, 2n - 3\} \rightarrow \{1, 2, 3, \ldots, 2n - 9\} \) by \( f(6) = 1 \) and \( f(x) = x - 6 \) for \( 8 \leq x \leq 2n - 3 \). Define \( F: \{\omega \in P_n | \omega_2 = 4\} \rightarrow P_{n-3} \) by \( \omega' = F(\omega) = f \circ \omega \circ f_6 \). \( F \) is a 1-1 correspondence such that \( r(\omega) = r(\omega') + 2 \) and \( \text{sgn}(\omega) = \text{sgn}(\omega') \) for each \( \omega \in P_n \) such that \( \omega_1 = 2, \omega_2 = 4 \), and \( \omega_3 = 5 \). Therefore, \( c_{n-3} = \sum (-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega' \in P_{n-3} \).

If \( \omega_1 = 2, \omega_2 = 4 \) and \( \omega_3 = 6 \), then \( v_5 \) and \( v_6 \) are connected by two subpaths of \( \omega \), namely \( e_8 \) and the subpath consisting of the edges \( e_7, e_3, e_1, e_5 \) and \( e_9 \). Either exactly two or exactly three of these six edges have orientations induced by \( \omega \) opposite to their given orientations. We may place the set of all \( \omega \in P_n \) such that \( \omega_1 = 2, \omega_2 = 4 \) and \( \omega_3 = 6 \) into 1-1 correspondence with itself by interchanging the order of these two subpaths if exactly 2 of the 6 edges have orientations induced by \( \omega \) opposite to their given orientations. We may place the set of all \( \omega \in P_n \) such that \( \omega_1 = 2, \omega_2 = 4 \) and \( \omega_3 = 6 \) into 1-1 correspondence with itself by interchanging the order of these two subpaths and reversing their induced orientations if exactly 3 of the 6 edges have orientations induced by \( \omega \) opposite to their given orientations. If \( \omega \leftrightarrow \omega' \) denotes this correspondence, then \( r(\omega) = r(\omega') \) and \( \text{sgn}(\omega) = -\text{sgn}(\omega') \). Therefore, \( \sum (-1)^{r(\omega')} \text{sgn}(\omega) = -\sum (-1)^{r(\omega')} \text{sgn}(\omega) \), and so \( \sum (-1)^{r(\omega')} \text{sgn}(\omega) = 0 \).
where each summation is over all \( \omega \in P_n \) such that \( \omega_1 = 2 \), \( \omega_2 = 4 \) and \( \omega_3 = 9 \).

We may place the set of all \( \omega \in P_n \) such that \( \omega_1 = 2 \), \( \omega_2 = 4 \) and \( \omega_3 = 9 \) into 1-1 correspondence with itself by reversing the induced orientation of the cycle consisting of the edges \( e_1, e_3, e_1, e_5 \) and \( e_6 \) for each \( \omega \in P_n \) such that \( \omega_1 = 2 \), \( \omega_2 = 4 \) and \( \omega_3 = 9 \). If \( \omega \leftrightarrow \omega' \) denotes this correspondence, then \( r(\omega) = r(\omega') + 1 \pmod{2} \) and \( \text{sgn}(\omega) = \text{sgn}(\omega') \). Therefore, \( \Sigma(-1)^{r(\omega)} \text{sgn}(\omega) = -\Sigma(-1)^{r(\omega')} \text{sgn}(\omega) \), and so \( \Sigma(-1)^{r(\omega')} \text{sgn}(\omega) = 0 \), where each summation is over all \( \omega \in P_n \) such that \( \omega_1 = 2 \), \( \omega_2 = 4 \) and \( \omega_3 = 9 \).

If \( \omega_1 = 2 \) and \( \omega_2 = 7 \), then there exists \( i, 4 \leq i \leq 2n - 7 \), such that

\[
\omega(i + j) \in \{1, 3, 4, 5\} \quad \text{for} \quad 0 \leq j \leq 3.
\]

Define \( f: \{6, 8, 9, \ldots, 2n - 3\} \rightarrow \{1, 2, 3, \ldots, 2n - 9\} \) by \( f(6) = 1 \) and \( f(x) = x - 6 \) for \( 8 \leq x \leq 2n - 3 \). Define \( \omega' = F(\omega) \) by \( \omega'(h) = f \circ \omega(h + 2) \) for \( 1 \leq h < i - 2 \) and \( \omega'(h) = f \circ \omega(h + 6) \) for \( i - 2 \leq h \leq 2n - 9 \). Define \( F: \{\omega \in P_n | \omega_1 = 2 \} \rightarrow P_{n-3} \) is a 2-1 map such that \( r(\omega) = r(\omega') + 2 \) and \( \text{sgn}(\omega) = \text{sgn}(\omega') \) for each \( \omega \in P_n \) such that \( \omega_1 = 2 \) and \( \omega_2 = 7 \). Therefore, \( -2c_{n-3} = -2\Sigma(-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega' \in P_{n-3} \), \( \Sigma(-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega' \in P_n \) such that \( \omega_1 = 2 \) and \( \omega_2 = 7 \).

**Proof of (e).** By direct computation \( \Sigma(-1)^{r(\omega')} \text{sgn}(\omega') = 0 \), where the summation is over all \( \omega \in P_n \) such that \( \omega_1 = 5 \). Hence, we may assume that

\[
n > 5, \quad \omega_2 = 4, 6 \text{ or } 9 \quad \text{for each} \quad \omega \in P_n \quad \text{such that} \quad \omega_1 = 5, \quad \omega_6 = 7 \quad \text{for each} \quad \omega \in P_n \quad \text{such that} \quad \omega_1 = 5 \quad \text{and} \quad \omega_2 = 4.
\]

Define \( f: \{6, 8, 9, \ldots, 2n - 3\} \rightarrow \{1, 2, 3, \ldots, 2n - 9\} \) by \( f(6) = 1 \) and \( f(x) = x - 6 \) for \( 8 \leq x \leq 2n - 3 \). Define \( F: \{\omega \in P_n | \omega_1 = 5 \} \rightarrow P_{n-3} \) by \( \omega' = F(\omega) = f \circ \omega \circ f \) and \( F \) is a 2-1 map such that \( r(\omega) = r(\omega') + 2 \) and \( \text{sgn}(\omega) = \text{sgn}(\omega') \) if \( \omega_3 = 3 \), \( \omega_4 = 1 \) and \( \omega_5 = 2 \), and \( r(\omega) = r(\omega') + 3 \) and \( \text{sgn}(\omega) = -\text{sgn}(\omega') \) if \( \omega_3 = 2 \), \( \omega_4 = 1 \) and \( \omega_5 = 3 \) for each \( \omega \in P_n \) such that \( \omega_1 = 5 \) and \( \omega_2 = 4 \). Therefore, \( 2c_{n-3} = 2\Sigma(-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega' \in P_{n-3} \), \( \Sigma(-1)^{r(\omega')} \text{sgn}(\omega') \), where the summation is over all \( \omega \in P_n \) such that \( \omega_1 = 5 \) and \( \omega_2 = 4 \).

If \( \omega_1 = 5 \) and \( \omega_2 = 6 \), then there exists \( i, 4 \leq i \leq 2n - 7 \), such that

\[
\omega(i + j) \in \{1, 2, 3, 4\} \quad \text{for} \quad 0 \leq j \leq 3.
\]

Define \( f: \{7, 8, 9, \ldots, 2n - 3\} \rightarrow \{1, 2, 3, \ldots, 2n - 9\} \) by \( f(x) = x - 6 \). Define \( \omega' = F(\omega) \) by \( \omega'(h) = f \circ \omega(h + 2) \) for \( 1 \leq h < i - 2 \) and \( \omega'(h) = f \circ \omega(h + 6) \) for \( i - 2 \leq h \leq 2n - 9 \). Define \( F: \{\omega \in P_n | \omega_1 = 5 \} \rightarrow P_{n-3} \) is a 2-1 map. If \( \omega_1 = 2 \), then \( r(\omega) = r(\omega') + 2 \) and \( \text{sgn}(\omega) = -\text{sgn}(\omega') \). If \( \omega_1 = 3 \), then \( r(\omega) = r(\omega') + 1 \) and \( \text{sgn}(\omega) = \text{sgn}(\omega') \). If \( \omega_1 = 4 \) and \( \omega(i + 1) = 2 \), then \( r(\omega) = r(\omega') + 3 \) and \( \text{sgn}(\omega) = \text{sgn}(\omega') \). If \( \omega_1 = 4 \) and \( \omega(i + 1) = 3 \), then \( r(\omega) = r(\omega') + 2 \) and \( \text{sgn}(\omega) = -\text{sgn}(\omega') \).
Thus, \((-1)^{r(\omega)} \text{sgn}(\omega) = -(-1)^{r(\omega')} \text{sgn}(\omega')\) for each \(\omega \in P_n\) such that \(\omega_1 = 5\) and \(\omega_2 = 6\). Therefore, \(-2c_{n-3} = -2 \sum (-1)^{r(\omega')} \text{sgn}(\omega')\), where the summation is over all \(\omega' \in P_{n-3}\), \(\sum (-1)^{r(\omega)} \text{sgn}(\omega)\), where the summation is over all \(\omega \in P_n\) such that \(\omega_1 = 5\) and \(\omega_2 = 6\).

We may place the set of all \(\omega \in P_n\) such that \(\omega_1 = 5\) and \(\omega_2 = 9\) into 1-1 correspondence with itself by reversing the induced orientation of the subpath consisting of the edges \(e_7, e_3, e_1, e_2, e_4\) and \(e_6\) for each \(\omega \in P_n\) such that \(\omega_1 = 5\) and \(\omega_2 = 9\). If \(\omega \leftrightarrow \omega'\) denotes this correspondence, then \(r(\omega) \equiv r(\omega') \pmod{2}\) and \(\text{sgn}(\omega) = -\text{sgn}(\omega')\). Therefore, \(\sum (-1)^{r(\omega)} \text{sgn}(\omega) = -\sum (-1)^{r(\omega')} \text{sgn}(\omega)\), and so \(\sum (-1)^{r(\omega)} \text{sgn}(\omega) = 0\), where each summation is over all \(\omega \in P_n\) such that \(\omega_1 = 5\) and \(\omega_2 = 9\).

The next lemma is immediate from Lemma 2 and Theorem 2.

**Lemma 3.** The \(c_n\)'s satisfy the difference equation \(c_n = c_{n-1} + c_{n-2} - c_{n-3}\) for \(n > 4\) with the initial conditions \(c_2 = c_3 = -2\) and \(c_4 = -6\).

We obtain by induction from Lemma 3 and \(c_{n-1} \leq c_{n-2} \leq c_{n-3}\) for \(n > 4\) that \(c_n = c_{n-1} + (c_{n-2} - c_{n-3}) \leq c_{n-1}\). Thus \(c_n \leq c_{n-1} < 0\) for \(n > 2\). In fact, solving the difference equation with the initial conditions in Lemma 3 we obtain \(c_n = 2 - 4\lambda(\frac{1}{2}n)\) for \(n > 1\). This completes the proof of Theorem 4.

As a final remark the computations above are all valid if the \(A_j\)'s are regarded as \(m \times m\) matrices for any \(m \leq n\).

**REFERENCES**


**MATHEMATICAL SCIENCES DEPARTMENT, BALL STATE UNIVERSITY, MUNCIE, INDIANA 47306**