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SMOOTH INTERPOLATING CURVES OF PRESCRIBED LENGTH AND MINIMUM CURVATURE[1]

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ABSTRACT. It is shown that, among all smooth curves of length not exceeding a prescribed upper bound which interpolate a finite set of planar points, there is at least one which minimizes the curvature in the L^2 sense. Thus, we show to be sufficient for the solution of the problem of minimum curvature a condition, viz., prescribed length, which has been known to be necessary for at least a decade. The proof extends immediately to curves in $\mathbf{R}^n$, $n > 2$.

Introduction. Garrett Birkhoff has conjectured the existence of a curve of minimum integral mean square curvature, interpolating a finite set of planar points, provided a feasible upper bound is prescribed for the length of the interpolating curve. The purpose of this note is to confirm this conjecture. It is dedicated to Professor Birkhoff on the occasion of his sixty-fourth birthday. That this condition of prescribed length is necessary was shown earlier by Birkhoff and de Boor [1]. It is now seen to be a natural necessary and sufficient condition for the existence of smooth interpolating curves of minimum curvature provided the point set is not collinear.

1. **Interpolating curves of minimum curvature.** Let $\mathcal{P} = \{(x_i, y_i)\}_{i=0}^N$ describe a set of points in $\mathbf{R}^2$ with $N \geq 1$ and let $L_0 > 0$ be given. Our basic hypothesis is the following.

(H) There is a function $\underline{r}(s) = (x(s), y(s))$, $0 \leq s \leq L$, such that

(i) $L \leq L_0$,

(ii) $(x, y) \in W^{2,2}(0, L) \times W^{2,2}(0, L)$,

(iii) $\mathcal{P} \subset \{(x(s), y(s)): 0 \leq s \leq L\}$,

(iv) $(dx/ds)^2 + (dy/ds)^2 = 1$, $0 \leq s \leq L$.

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Hypothesis (H) insures that there is a smooth interpolating curve of length L, not exceeding L_0, parametrized by arc length with a well-defined (a.e.) square integrable curvature

$$\kappa(s) = \{(d^2x/ds^2)^2 + (d^2y/ds^2)^2\}^{1/2} = |d\theta/ds|$$

where θ is the usual polar angle. Here $W^{2,2}(0, L)$ represents the real Sobolev class of functions f such that f, f' are absolutely continuous on $[0, L]$ with f'' square integrable. Our fundamental existence result is the following.

Theorem. *Let U denote the set of vector functions satisfying hypothesis* (H). *Then there is at least one vector function whose graph is of minimum L^2-curvature, i.e., for which $\{\int_0^L \kappa^2(s)\,ds\}^{1/2}$ is a minimum.*

Remark. A related result, in which the curves were plane graphs of real functions defined on a compact interval, was obtained by the writer in [3]. The hypothesis of curve length not exceeding L_0 is a localizing condition. A global minimum is not achievable, in general, as pointed out by Birkhoff and de Boor [1], if no restriction is made on the lengths of the interpolating curves. Notice that, in the Theorem, we do not require the curve to pass through the points of $\mathcal{P}$ in any prescribed order.

Proof of Theorem. Let $\mathfrak{W} = W^{2,2}(0, L_0) \times W^{2,2}(0, L_0)$. $\mathfrak{W}$ is a Hilbert space with inner product

$$(\underline{f}, \underline{h}) = \int_0^{L_0} \sum_{i=1}^{2} \sum_{j=0}^{2} D^j f_i(s) D^j h_i(s)\,ds.$$

Every function $\underline{r}$ satisfying (H) can be identified with a member of $\mathfrak{W}$ with the same mean square curvature; x and y are extended as linear functions of s to $L \le s \le L_0$. The mapping which embeds U into $\mathfrak{W}$ is denoted by J. Note that, for every $\underline{f} \in JU$, $(df_1/ds)^2 + (df_2/ds)^2 = 1$ for $0 \le s \le L_0$.

Now define the mapping $T\colon \mathfrak{W} \to L^2(0, L_0) \times L^2(0, L_0)$ by $T\underline{f} = (D^2 f_1, D^2 f_2)$. If $\alpha = \inf\{\|T\underline{f}\|_{L^2 \times L^2} : \underline{f} \in JU\}$, where

$$\|T\underline{f}\|^2_{L^2 \times L^2} = \|D^2 f_1\|^2_{L^2} + \|D^2 f_2\|^2_{L^2},$$

then we seek a function $\underline{F} \in JU$ such that

$$\|T\underline{F}\|_{L^2 \times L^2} = \left\{\int_0^L \kappa^2(s)\,ds\right\}^{1/2} = \alpha.$$

The verification of the existence of such an $\underline{F}$ requires three stages: JU is weakly closed in $\mathfrak{W}$; T maps weakly convergent sequences onto similar such sequences; and, every minimizing sequence is bounded in $\mathfrak{W}$.

To see that JU is weakly closed, let $\{\underline{f}_n\} \subset JU$ with $\underline{f}_n \rightharpoonup \underline{f} \in \mathfrak{W}$. Notice that f_{ni} and Df_{ni} converge uniformly to f_i and Df_i respectively, $i = 1, 2$, since the injection $\mathfrak{W} \to C^1[0, L_0] \times C^1[0, L_0]$ is compact and the image of a weakly convergent sequence under a compact mapping is convergent. Clearly, then

$$(df_1/ds)^2 + (df_2/ds)^2 = 1 \quad \text{for } 0 \le s \le L_0 \tag{1}$$

since a corresponding relation holds for each $\underline{f}_n$. It remains to show that $\mathcal{P} \subset \{(f_1(s), f_2(s)) \colon 0 \le s \le L_0\}$. Thus, fix a point $(x_*, y_*) \in \mathcal{P}$. For the $n = 1, 2, \cdots$ let $s_n \in [0, L_0]$ satisfy $\underline{f}_n(s_n) = (x_*, y_*)$.

If s_* is an accumulation point of $\{s_n\}$ then $\underline{f}(s_*) = (x_*, y_*)$. Indeed, let $\epsilon > 0$ and, by the equicontinuity of $\{f_{n1}\}$ on $[0, L_0]$, select $\delta > 0$ such that $|s - t| < \delta \Rightarrow |f_{n1}(s) - f_{n1}(t)| < \epsilon/2$ for each $n = 1, 2, \cdots$. Then, if n_0 satisfies $|s_{n_0} - s_*| < \delta$ and $|f_1(s_*) - f_{n_0 1}(s_*)| < \epsilon/2$, we have

$$|f_1(s_*) - x_*| \le |f_1(s_*) - f_{n_0 1}(s_*)| + |f_{n_0 1}(s_*) - f_{n_0 1}(s_{n_0})| < \epsilon.$$

Thus $f_1(s_*) = x_*$ and a similar argument shows that $f_2(s_*) = y_*$. We conclude that JU is weakly closed.

Suppose now that $\underline{f}_n \rightharpoonup \underline{f}$ in $\mathfrak{W}$. We show that $T\underline{f}_n \rightharpoonup T\underline{f}$ in $L^2(0, L_0) \times L^2(0, L_0)$. Let $\underline{\phi} = (\phi_1, \phi_2) \in L^2(0, L_0) \times L^2(0, L_0)$. It suffices to show $(T\underline{f}_n, \underline{\phi})_{L^2 \times L^2} \to (T\underline{f}, \underline{\phi})_{L^2 \times L^2}$ as $n \to \infty$. Select $\underline{\psi} \in \mathfrak{W}$ such that $T\underline{\psi} = \underline{\phi}$. Then, using the notation $D\underline{f} = (Df_1, Df_2)$ we have

$$\begin{aligned} &(\underline{f}_n, \underline{\psi})_{L^2 \times L^2} + (D\underline{f}_n, D\underline{\psi})_{L^2 \times L^2} + (T\underline{f}_n, T\underline{\psi})_{L^2 \times L^2} \\ &\qquad \to (\underline{f}, \underline{\psi})_{L^2 \times L^2} + (D\underline{f}, D\underline{\psi})_{L^2 \times L^2} + (T\underline{f}, T\underline{\psi})_{L^2 \times L^2} \end{aligned}$$

since $\underline{f}_n \rightharpoonup \underline{f}$. Now by the aforementioned uniform convergence of $\underline{f}_n$ to $\underline{f}$ and $D\underline{f}_n$ to $D\underline{f}$, it follows that $T\underline{f}_n \rightharpoonup T\underline{f}$.

Finally, suppose that $\underline{f}_n$ is a minimizing sequence, i.e., $\{\underline{f}_n\} \subset JU$ and

$$\|T\underline{f}_n\|_{L^2 \times L^2} \to \alpha \quad \text{as } n \to \infty. \tag{2}$$

Then $\{\underline{f}_n\}$ is bounded in $\mathfrak{W}$. Indeed,

$$(D\underline{f}_n, D\underline{f}_n)_{L^2 \times L^2} = L_0, \qquad n = 1, 2, \cdots, \tag{3}$$

by (1). Also, for $n = 1, 2, \cdots$

$$f_{n1}(s) = x_0 + \int_{s_n}^{s} f'_{n1}(\sigma)\, d\sigma, \qquad f_{n2}(s) = y_0 + \int_{s_n}^{s} f'_{n2}(\sigma)\, d\sigma$$

so that

$$\max_{0<s<L_0} \{|f_{n1}(s)|^2 + |f_{n2}(s)|^2\} \leq 2(|x_0|^2 + |y_0|^2 + 2L_0^2). \tag{4}$$

It follows from (2), (3) and (4) that $\{\underline{f}_n\}$ is bounded in $\mathfrak{W}$.

Theorem 1 is now a consequence of the following lemma, quoted from [3].

Lemma 1. *Let X be a reflexive Banach space, Y a normed linear space and U a weakly closed subset of X. Let T be a mapping with the property that Tx_n is weakly convergent to Tx in Y if x_n is weakly convergent to x in X. Then, if there exists a bounded sequence $\{x_n\} \subset U$ such that $\|Tx_n\|_Y \to \inf\{\|Tx\|_Y : x \in U\} = \alpha$, there is an element $x_0 \in U$ satisfying $\|Tx_0\|_Y = \alpha$.*

Remark. In the case of graphs of interpolating functions on a compact interval, it was shown via the Gâteaux derivative that a solution of minimum continuous curvature exists locally satisfying the differential equation [3] $d^2\kappa/ds^2 + \kappa^3/2 = 0$. Forsythe and Lee [2] have carried out a formal calculus of variations for the nonlinear (open) spline curves considered here, requiring only that the spline curves pass through the points of $\mathcal{P}$ consecutively. As necessary conditions, they deduce the continuity of the curvature as well as the (local) defining differential equation $(d^2\kappa/ds^2 + \kappa^{3/2}) = 0$. For the deduction of these conditions by function calculus methods for a solution $\underline{F}$, guaranteed by Theorem 1, cf. [4]. We mention finally that Theorem 1 is valid for curves in $\mathbf{R}^n$, $n > 2$. The curvature, in this case, is the usual expression

$$\kappa(s) = |(d/ds)(dx_1/ds, \cdots, dx_n/ds)|$$

where the curve $\underline{x} = \underline{x}(s)$ is parametrized by arc length.

Remark added in proof. It has been brought to the author's attention that the failure of the existence of a global minimum is mentioned also (in addition to [1]) in the technical report, *Nonlinear interpolation by splines, pseudosplines and elastica*, authored by G. Birkhoff, H. Burchard and D. Thomas as General Motors Research Publication 468, Warren, Michigan, February 3, 1965.

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