ON THE NUMBER OF IRREDUCIBLE REPRESENTATIONS
OF DEGREE \( \leq n \) OF A LIE GROUP

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ABSTRACT. We give a proof of part of a result of Robert Cahn [1] on the asymptotic behavior of the number of irreducible representations of degree \( \leq n \) of a semisimple Lie group. The argument is a general one and does not depend on classification.

Let \( \mathfrak{g} \) be a complex simple Lie algebra of rank \( a \) with \( b \) positive roots. Let \( \alpha_1, \alpha_2, \cdots, \alpha_a \) be a simple system of roots, and \( \lambda_1, \lambda_2, \cdots, \lambda_a \) the weights of the fundamental representations. If \( \lambda = \sum_{i=1}^{a} z_i \lambda_i \), \( z_i \) nonnegative integers, is the dominant weight of an irreducible representation \( \pi \), then from the formula of Weyl,

\[
\text{degree of } \pi = \prod_{\alpha > 0} (\lambda + \delta, \alpha) / \prod_{\alpha > 0} (\delta, \alpha)
\]

where \( \delta \) is half the sum of the positive roots.

Let \( x \in \mathbb{R}^a \), and put \( X = \sum x_i \lambda_i \). A recent paper by Robert Cahn [1] gives an asymptotic formula for the number of irreducible representations of degree \( \leq n \), the proof of which depends first on showing that for the polynomial \( W(x) = \prod_{\alpha > 0} (X, \alpha) \), the volume of the region \( S = \{ x \in \mathbb{R}^a | x_i \geq 0, W(X) \leq 1 \} \) is finite. The proof of the latter follows essentially from a case by case study of the simple algebras, the verification for the exceptional algebras being left in fact to the reader.

In this note we give a general proof based in part on a recent short note of ours [2], and for the rest on well-known properties of Dynkin diagrams, and not requiring any case by case considerations.

First we describe the relevant result from [2]. Let polynomial \( P(x) \), \( x \in \mathbb{R}^a \), be a product of \( b \) linear forms:

\[
P(x) = \prod_{\nu=1}^{b} (c_{\nu,1} x_1 + c_{\nu,2} x_2 + \cdots + c_{\nu,a} x_a),
\]
each linear form having nonnegative coefficients not all zero. Let \( U \) be a subset of \( \{1, 2, \cdots, a\} \). We say that the support of the linear form \( c_1 x_1 + c_2 x_2 + \cdots + c_a x_a \) is \( U \) if \( c_i \neq 0 \) for \( i \in U \) and \( c_i = 0 \) for \( i \notin U \). For any subset \( U \), let \( N(U) \) be the number of linear forms in product for \( P(x) \) whose supports are contained in \( U \). Then we have

**Lemma 1.** Let \( S = \{x \in \mathbb{R}^a | x_i \geq 0, P(x) \leq 1\} \). The volume of \( S \) is finite if and only if for every proper subset \( U \) we have \( N(U)/\text{card } U < b/a \).

Next we apply the last result to the polynomial \( W(X) \) above. Let \( U \) be a subset of \( \{1, 2, \cdots, a\} \). Associate to \( U \) the subset of roots \( \{\alpha_i | i \in U\} = U(\alpha) \). Let \( \mathfrak{g}(U) \) be the smallest Lie subalgebra containing all the root vectors \( X_{\alpha_i}, X_{-\alpha_i}, i \in U \). \( \mathfrak{g}(U) \) is a semisimple algebra whose Dynkin diagram is obtained from that of \( \mathfrak{g} \) by suppressing the simple roots not in \( U(\alpha) \) and the lines issuing therefrom. We say that \( U \) is connected if \( \mathfrak{g}(U) \) is simple. This means the suppressed diagram is connected. What we have to prove is that the number of positive roots which are linear combinations of the members of \( U(\alpha) \), i.e., the number of positive roots with support contained in \( U \), is smaller than \( b/a \text{ card } U \) for every proper subset \( U \). As before let \( N(U) \) be the number of positive roots with support contained in \( U \). Every root with support contained in \( U \) is a root of the Lie algebra \( \mathfrak{g}(U) \). If \( U \) is not connected, it may be partitioned into two disjoint proper subsets of \( U \), \( U = U_1 \oplus U_2 \), with no connections between them. This means \( (\alpha_i, \alpha_j) = 0 \) if \( i \in U_1, j \in U_2 \). It follows that every positive root supported in \( U \) is supported in either \( U_1 \) or \( U_2 \), so \( N(U) = N(U_1) + N(U_2) \). First we prove

**Lemma 2.** \( 1/2 + N(1, 2, \cdots, a - 1)/(a - 1) \leq N(1, 2, \cdots, a)/a = b/a \) \( (a \geq 2) \).

The proof is by induction on \( a \). For \( a = 2 \), there is little to prove, so we take \( a \geq 3 \). Suppose first \( \{1, 2, \cdots, a - 1\} \) is connected. \( \alpha_a \) must be connected to at least one of \( \alpha_1, \alpha_2, \cdots, \alpha_{a-1} \), say \( \alpha_{a-1} \), so \( (\alpha_{a-1}, \alpha_a) < 0 \). By our induction hypothesis

\[
N(1, 2, \cdots, a - 2)/(a - 2) \leq N(1, 2, \cdots, a - 1)/(a - 1) - \frac{1}{2}.
\]

Consider the \( N = N(1, 2, \cdots, a - 1) - N(1, 2, \cdots, a - 2) \) positive roots whose support contains \( \{a - 1\} \). Apply to each of them the Weyl reflection \( W_{\alpha_a} \). We obtain \( N \) more positive roots whose support contains \( \{a\} \). And we have in addition the root \( \alpha_a \) itself, not counted among the additional \( N \).

Hence
\[
\begin{align*}
\frac{N(1, 2, \cdots, a)}{a} &\geq \frac{N(1, 2, \cdots, a-2) + 2N + 1}{a} \\
&= \frac{2N(1, 2, \cdots, a-1) - N(1, 2, \cdots, a-2)}{a} + \frac{1}{a} \\
&\geq \frac{2N(1, 2, \cdots, a-1) - [(a-2)/(a-1)]N(1, 2, \cdots, a-1) + (a-2)/2 + 1}{a} \\
&= \frac{N(1, 2, \cdots, a-1)}{a-1} + \frac{1}{2}
\end{align*}
\]

which completes induction in this case.

Next suppose \( U = \{1, 2, \cdots, a-1\} \) is not connected, so \( a \geq 3 \). Partition \( U \) into proper disjoint connected sets \( U_1, U_2, \cdots, U_k \) with no connections between sets. Then \( N(U) = N(U_1) + N(U_2) + \cdots + N(U_k) \). And \( a \) must be connected to some root in each of \( U_1, U_2, \cdots, U_k \), since the whole diagram is connected. Let \( C_v = \text{card } U_v \). Applying inductive hypothesis again, we have

\[
N(U_v, a) \geq \frac{1 + C_v}{2} + \frac{1 + C_v}{C_v} N(U_v).
\]

Thus

\[
\frac{N(1, 2, \cdots, a)}{a} \geq \frac{N(U_1, a) + N(U_2, a) + \cdots + N(U_k, a) - (k-1)}{a}
\]

(where term \(- (k-1)\) takes care of multiple counting of root \( a \))

\[
\begin{align*}
&\geq \frac{a+1-k}{2a} + \frac{1}{a} \sum_{v=1}^{k} \frac{1 + C_v}{C_v} N(U_v) \\
&= \frac{a+1-k}{2a} + \frac{N(U)}{a-1} + \sum_{v=1}^{k} \left( \frac{1 + C_v}{ac_v} - \frac{1}{a-1} \right) N(U_v) \\
&= \frac{a+1-k}{2a} + \frac{N(U)}{a-1} + \sum_{v=1}^{k} \frac{a-1-C_v}{a(a-1)c_v} N(U_v) \\
&\geq \frac{a+1-k}{2a} + \frac{N(U)}{a-1} + \sum_{v=1}^{k} \frac{a-1-C_v}{a(a-1)} \\
&= \frac{N(U)}{a-1} + \frac{1}{2} + \frac{k-1}{2a} \geq \frac{N(U)}{a-1} + \frac{1}{2},
\end{align*}
\]

where we have used the obvious fact that \( N(U_v) \geq C_v \) to complete the induction in this case.
Now for the general proposition:

Theorem 3. \( b/a > N(1, 2, \ldots, r)/r + (a - r)/2 \).

Proof. First suppose \( U = \{1, 2, \ldots, r\} \) is connected. Order the remaining roots so that always \( \{1, 2, \ldots, s\} \) is connected, \( r < s < a \). By the last lemma

\[
\frac{N(1, 2, \ldots, r)}{r} \leq \frac{N(1, 2, \ldots, r+1)}{r+1} - \frac{1}{2} \leq \cdots \leq \frac{b}{a} - \frac{a-r}{2}.
\]

If \( U = \{1, 2, \ldots, r\} \) is not connected, partition \( U \) into proper disjoint connected sets \( U_1, U_2, \ldots, U_k \) with no connections between sets, and of respective cardinalities \( C_1, C_2, \ldots, C_k \). By the argument immediately above,

\[
N(U_\nu) / C_\nu \leq b/a - (a - C_\nu)/2.
\]

So

\[
N(U) = \sum_{\nu=1}^{k} N(U_\nu) \leq \frac{b}{a} \sum_{\nu=1}^{k} C_\nu - \frac{1}{2} \sum_{\nu=1}^{k} C_\nu (a - C_\nu)
\]

\[
= \frac{r}{a} - \frac{1}{2} ar + \frac{1}{2} \sum_{\nu=1}^{k} C_\nu^2 \leq \frac{b}{a} - \frac{1}{2} ar + \frac{r^2}{2} = \frac{r}{a} - r \frac{(a-r)}{2}
\]

completing proof of the theorem, and showing moreover that the polynomial \( W(\lambda) \) easily satisfies hypotheses of Lemma 1.

REFERENCES