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ON THE NUMBER OF IRREDUCIBLE REPRESENTATIONS OF DEGREE $\leq n$ OF A LIE GROUP

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ABSTRACT. We give a proof of part of a result of Robert Cahn [1] on the asymptotic behavior of the number of irreducible representations of degree $\leq n$ of a semisimple Lie group. The argument is a general one and does not depend on classification.

Let $\mathfrak{G}$ be a complex simple Lie algebra of rank a with b positive roots. Let $\alpha_1, \alpha_2, \cdots, \alpha_a$ be a simple system of roots, and $\lambda_1, \lambda_2, \cdots, \lambda_a$ the weights of the fundamental representations. If $\lambda = \sum_1^a z_i\lambda_i$, z_i nonnegative integers, is the dominant weight of an irreducible representation π, then from the formula of Weyl,

$$\text{degree of } \pi = \prod_{\alpha>0} (\lambda + \delta, \alpha) \Big/ \prod_{\alpha>0} (\delta, \alpha)$$

where δ is half the sum of the positive roots.

Let $x \in \mathbf{R}^a$, and put $X = \sum x_i\lambda_i$. A recent paper by Robert Cahn [1] gives an asymptotic formula for the number of irreducible representations of degree $\leq n$, the proof of which depends first on showing that for the polynomial $W(x) = \prod_{\alpha>0} (X, \alpha)$, the volume of the region $S = \{x \in \mathbf{R}^\alpha | x_i \geq 0, W(X) \leq 1\}$ is finite. The proof of the latter follows essentially from a case by case study of the simple algebras, the verification for the exceptional algebras being left in fact to the reader.

In this note we give a general proof based in part on a recent short note of ours [2], and for the rest on well-known properties of Dynkin diagrams, and not requiring any case by case considerations.

First we describe the relevant result from [2]. Let polynomial $P(x)$, $x \in \mathbf{R}^a$, be a product of b linear forms:

$$P(x) = \prod_{v=1}^{b} (c_{v,1}x_1 + c_{v,2}x_2 + \cdots + c_{v,a}x_a),$$

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each linear form having nonnegative coefficients not all zero. Let U be a subset of $\{1, 2, \cdots, a\}$. We say that the support of the linear form $c_1x_1 + c_2x_2 + \cdots + c_ax_a$ is U if $c_i \neq 0$ for $i \in U$ and $c_i = 0$ for $i \notin U$. For any subset U, let $N(U)$ be the number of linear forms in product for $P(x)$ whose supports are contained in U. Then we have

Lemma 1. *Let $S = \{x \in \mathbf{R}^a | x_i \geq 0, P(x) \leq 1\}$. The volume of S is finite if and only if for every proper subset U we have $N(U)/\text{card } U < b/a$.*

Next we apply the last result to the polynomial $W(X)$ above. Let U be a subset of $\{1, 2, \cdots, a\}$. Associate to U the subset of roots $\{\alpha_i | i \in U\} = U(\alpha)$. Let $\mathfrak{G}(U)$ be the smallest Lie subalgebra containing all the root vectors $X_{\alpha_i}, X_{-\alpha_i}$, $i \in U$. $\mathfrak{G}(U)$ is a semisimple algebra whose Dynkin diagram is obtained from that of $\mathfrak{G}$ by suppressing the simple roots not in $U(\alpha)$ and the lines issuing therefrom. We say that U is connected if $\mathfrak{G}(U)$ is simple. This means the suppressed diagram is connected. What we have to prove is that the number of positive roots which are linear combinations of the members of $U(\alpha)$, i.e., the number of positive roots with support contained in U, is smaller than b/a card U for every proper subset U. As before let $N(U)$ be the number of positive roots with support contained in U. Every root with support contained in U is a root of the Lie algebra $\mathfrak{G}(U)$. If U is not connected, it may be partitioned into two disjoint proper subsets of U, $U = U_1 \oplus U_2$, with no connections between them. This means $(\alpha_i, \alpha_j) = 0$ if $i \in U_1$, $j \in U_2$. It follows that every positive root supported in U is supported in either U_1 or U_2, so $N(U) = N(U_1) + N(U_2)$. First we prove

Lemma 2. $\frac{1}{2} + N(1, 2, \cdots, a-1)/(a-1) \leq N(1, 2, \cdots, a)/a = b/a$ $(a \geq 2)$.

The proof is by induction on a. For $a = 2$, there is little to prove, so we take $a \geq 3$. Suppose first $\{1, 2, \cdots, a-1\}$ is connected. α_a must be connected to at least one of $\alpha_1, \alpha_2, \cdots, \alpha_{a-1}$, say α_{a-1}, so $(\alpha_{a-1}, \alpha_a) < 0$. By our induction hypothesis

$$N(1, 2, \cdots, a-2)/(a-2) \leq N(1, 2, \cdots, a-1)/(a-1) - \tfrac{1}{2}.$$

Consider the $N = N(1, 2, \cdots, a-1) - N(1, 2, \cdots, a-2)$ positive roots whose support contains "$a-1$". Apply to each of them the Weyl reflection W_{α_a}. We obtain N more positive roots whose support contains "a". And we have in addition the root α_a itself, not counted among the additional N. Hence

$$\frac{N(1, 2, \cdots, a)}{a} \geq \frac{N(1, 2, \cdots, a-2) + 2N + 1}{a}$$

$$= \frac{2N(1, 2, \cdots, a-1) - N(1, 2, \cdots, a-2)}{a} + \frac{1}{a}$$

$$\geq \frac{2N(1, 2, \cdots, a-1) - [(a-2)/(a-1)]N(1, 2, \cdots, a-1) + (a-2)/2 + 1}{a}$$

$$= \frac{N(1, 2, \cdots, a-1)}{a-1} + \frac{1}{2}$$

which completes induction in this case.

Next suppose $U = \{1, 2, \cdots, a-1\}$ is not connected, so $a \geq 3$. Partition U into proper disjoint connected sets $U_1, U_2, \cdots, U_k$ with no connections between sets. Then $N(U) = N(U_1) + N(U_2) + \cdots + N(U_k)$. And α_a must be connected to some root in each of $U_1, U_2, \cdots, U_k$, since the whole diagram is connected. Let C_v = card U_v. Applying inductive hypothesis again, we have

$$N(U_v, a) \geq \frac{1 + C_v}{2} + \frac{1 + C_v}{C_v} N(U_v).$$

Thus

$$\frac{N(1, 2, \cdots, a)}{a} \geq \frac{N(U_1, a) + N(U_2, a) + \cdots + N(U_k, a) - (k-1)}{a}$$

(where term $-(k-1)$ takes care of multiple counting of root α_a)

$$\geq \frac{a+1-k}{2a} + \frac{1}{a} \sum_{v=1}^{k} \frac{1 + C_v}{C_v} N(U_v)$$

$$= \frac{a+1-k}{2a} + \frac{N(U)}{a-1} + \sum_{v=1}^{k} \left(\frac{1 + C_v}{aC_v} - \frac{1}{a-1} \right) N(U_v)$$

$$= \frac{a+1-k}{2a} + \frac{N(U)}{a-1} + \sum_{v=1}^{k} \frac{a-1-C_v}{a(a-1)C_v} N(U_v)$$

$$\geq \frac{a+1-k}{2a} + \frac{N(U)}{a-1} + \sum_{v=1}^{k} \frac{a-1-C_v}{a(a-1)}$$

$$= \frac{N(U)}{a-1} + \frac{1}{2} + \frac{k-1}{2a} \geq \frac{N(U)}{a-1} + \frac{1}{2},$$

where we have used the obvious fact that $N(U_v) \geq C_v$ to complete the induction in this case.

Now for the general proposition:

Theorem 3. $b/a \geq N(1, 2, \cdots, r)/r + (a - r)/2$.

Proof. First suppose $U = \{1, 2, \cdots, r\}$ is connected. Order the remaining roots so that always $\{1, 2, \cdots, s\}$ is connected, $r \leq s \leq a$. By the last lemma

$$\frac{N(1, 2, \cdots, r)}{r} \leq \frac{N(1, 2, \cdots, r+1)}{r+1} - \frac{1}{2} \leq \cdots \leq \frac{b}{a} - \frac{a-r}{2}.$$

If $U = \{1, 2, \cdots, r\}$ is not connected, partition U into proper disjoint connected sets $U_1, U_2, \cdots, U_k$ with no connections between sets, and of respective cardinalities $C_1, C_2, \cdots, C_k$. By the argument immediately above,

$$N(U_v)/C_v \leq b/a - (a - C_v)/2.$$

So

$$N(U) = \sum_{v=1}^{k} N(U_v) \leq \frac{b}{a} \sum_{v=1}^{k} C_v - \frac{1}{2} \sum_{v=1}^{k} C_v(a - C_v)$$

$$= r\frac{b}{a} - \frac{1}{2} ar + \frac{1}{2} \sum_{v=1}^{k} C_v^2 \leq r\frac{b}{a} - \frac{1}{2} ar + \frac{r^2}{2} = r\frac{b}{a} - r\frac{(a-r)}{2}$$

completing proof of the theorem, and showing moreover that the polynomial $W(X)$ easily satisfies hypotheses of Lemma 1.

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