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# ON INFINITELY DIVISIBLE LAWS IN $C[0, 1]$

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ABSTRACT. In Euclidean spaces, or in a separable Hilbert space, the characteristic function of an infinitely divisible distribution has the familiar form given by the Lévy-Khintchine formula. The Lévy measures $M$ of this formula are characterized by the property that the integral of $\min[1, \|x\|^2]$ with respect to $M$ is finite. This simple situation no longer holds in the Banach space $C = C[0, 1]$ where integrability of $\min[1, \|x\|]$ is sufficient but integrability of $\min[1, \|x\|^2]$ is neither necessary nor sufficient.

Certain other conditions which are sufficient to imply that $M$ is the Lévy measure of a distribution on $C$ can be obtained with the use of an integral formula of Garsia.

1. **Introduction.** Let $C$ be the Banach space $C = C[0, 1]$ of continuous functions on the interval $[0, 1]$ with the uniform norm. Let $M$ be a positive measure on $C$. Suppose that for each element $\mu$ of the dual $C'$ of $C$ the integral $\int \min\{1, |\langle \mu, x\rangle|^2\} M(dx)$ is finite. Let $h$ be the function

$$h(\mu, x) = \exp\{i\langle \mu, x\rangle\} - 1 - i\langle \mu, x\rangle[1 + |\langle \mu, x\rangle|^2]^{-1}\}. \tag{1}$$

Define $\psi$ on $C'$ by

$$\psi(\mu) = \exp \int h(\mu, x) M(dx). \tag{2}$$

It can be shown that such a $\psi$ is always the Fourier transform of an infinitely divisible distribution $P$ carried by the algebraic dual of $C'$. We shall call $M$ a proper Lévy measure if the probability measure $P$ is carried by $C$ itself. The literature does not seem to contain any proof of the fact that every infinitely divisible distribution $P$ carried by $C$ is the convolution of a Gaussian measure by a measure whose Fourier transform has the form (2). This is true, as we shall show elsewhere, and is well known for certain special Banach spaces other than $C$. For instance, if $C$ were replaced by a separable Hilbert space, one would know from Parthasarathy [1] that infinitely divisible distributions have a Lévy-Khintchine representation. In addition, we would know from Varadhan [2] that a necessary and sufficient

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condition for a positive measure $M$ to be a proper Lévy measure is that $\int \min[1, \|x\|^2] M(dx) < \infty$.

We shall proceed to show, in § 2 below, that $\int \min[1, \|x\|] M(dx) < \infty$ is always a sufficient condition, but give counterexamples showing that in $C$ the integrability of $\min[1, \|x\|^2]$ is neither necessary nor sufficient. Extensions to functions of the norm which decrease faster than $\|x\|$ are possible and will be given in another publication. This shows that integrability conditions on functions of the norm are not necessarily the appropriate criteria. We proceed in § 3 to give certain conditions which are sufficient to imply that $M$ is a proper Lévy measure, using a formula of Garsia, Rodemich and Rumsey [3]. The results of this nature can also be rephrased in terms of entropy conditions such as used in the Gaussian case by Dudley [4]. They lead naturally to sufficient conditions for central limit theorems in $C[0, 1]$.

2. **Integrability of functions of the norm.** Let $M$ be a positive measure in a separable Banach space. Necessary and sufficient conditions for $M$ to be a proper Lévy measure, that is, the Lévy measure of a distribution carried by the Banach space, are known in the finite dimensional or Hilbertian case where they take the form $\int \min[1, \|x\|^2] M(dx) < \infty$. We first show that such a condition is neither necessary nor sufficient in $C = C[0, 1]$. The examples will use two sequences $\{X_n\}$ and $\{X_n'\}$ of independent Poisson random variables such that $EX_n = \lambda_n = EX_n'$, defined on a probability space $\Omega$, and sequences $\{g_n\}$, $g_n \in C$, $\|g_n\| \leq 1$. When the series converges in probability for each $\mu \in C'$, the sum

$$\langle \mu, Y\rangle = \sum_n (X_n - X_n')\langle \mu, g_n\rangle \tag{3}$$

defines a stochastic process. Its characteristic function has the form

$$E \exp\{i\langle \mu, Y\rangle\} = \exp \int [\exp\{i\langle \mu, x\rangle\} - 1] M(dx) \tag{4}$$

for a Lévy measure $M$ which assigns masses $\lambda_n$ to each of $\{g_n\}$ and $\{-g_n\}$ for each $n$.

**Proposition 1.** *In the space $C$, there are sequences $\{\lambda_n, g_n\}$ such that $M$, defined above, is a proper Lévy measure, but* $\int \min[1, \|x\|^2] M(dx) = \infty$.

**Proof.** Let $\lambda_n = n^{-3/4}$ and $c_n = n^{-1/8}$. Let $f_n$ be the function defined by $f_n[3 \cdot 2^{-(n+1)}] = 1$, $f_n(s) = 0$ if $s \notin I_n = (2^{-n}, 2^{-(n-1)})$, and linear interpolation in between. Take for $g_n$ in formula (3) the function $g_n = c_n f_n$. Then $\int \|x\|^2 M(dx) = 2 \sum c_n^2 \lambda_n = \infty$.

To show that $M$ is nevertheless a proper Lévy measure consider the sum $Y(s, \omega) = \Sigma_n [X_n(\omega) - X'_n(\omega)] g_n(s)$. For $s > 0$ this is continuous since it is a finite sum. Also

$$\sum P\{|X_n - X'_n| > 1\} \leq 2 \sum P[X_n > 1] \leq 4 \sum \lambda_n^2 < \infty.$$

Thus, according to the Borel-Cantelli theorem, for almost every $\omega \in \Omega$ there is an integer $n(\omega)$ such that $n \geq n(\omega)$ implies $|X_n(\omega) - X'_n(\omega)| \leq 1$. For $n \geq n(\omega)$ and $s \in I_n$ this gives the inequality $|Y(s, \omega)| \leq |X_n(\omega) - X'_n(\omega)| g_n(s) \leq n^{-1/8}$. Since $Y(s, \omega) = 0$, this implies the almost sure continuity of $Y$ and the convergence of the series which defines $\langle \mu, Y \rangle$, concluding the proof of the proposition.

**Proposition 2.** *There are sequences* $\{\lambda_n, g_n\}$ *such that* $Y(s, \omega) = \Sigma [X_n(\omega) - X'_n(\omega)] g_n(s)$ *is almost surely an unbounded function of* $s$, *even though* $\int \|x\|^2 M(dx) < \infty$.

**Proof.** Let $\lambda_n = 1$, $n = 1, 2, \ldots$. For $j = 1, 2, \ldots$, let $a_n$ be equal to $j^{-1} 2^{-j/2}$ if $2^j \leq n < 2^{j+1}$. Take for function $g_n$ the expression $g_n(s) = a_n \cos(2\pi n s + b_n)$. Then $\int \|x\|^2 M(dx) = \Sigma_j j^{-2} < \infty$. Also, the series defining $\langle \mu, Y \rangle$ converges in quadratic mean since $\Sigma a_n^2 < \infty$. We claim that the constant $b_n$ may be chosen so that for almost all $\omega$ one has $\sup_s |Y(s, \omega)| = \infty$. To prove this note that if $s_j^2 = \Sigma_n \{a_n^2;\ 2^j \leq n < 2^{j+1}\}$ then $\Sigma_j s_j = \infty$ and consider the sums

$$Z(s, \omega) = \sum_n a_n [X_n(\omega) - X'_n(\omega)] \cos[2\pi n s + \phi_n(\omega)]$$

where the $\phi_n(\omega)$ are uniformly distributed on $[0, 2\pi)$ independently of each other and of the sequences $\{X_n\}$, $\{X'_n\}$. A lemma of Paley and Zygmund (see for instance Kahane [5, p. 85]) says that the function $s \to Z(s, \omega)$ is almost surely not an element of $L^\infty$. By Fubini's theorem this remains true for almost all values of the $\phi_n(\omega)$ implying the existence of sequences $\{b_n\}$ for which $s \to Y(s, \omega)$ is almost surely not an element of $L^\infty$.

In such a case $M$ is a Lévy measure for a process on the algebraic dual of $C'$, but not for a distribution on $C$ or even on the second dual $C''$.

LeCam suggested (private communication) that the construction of Proposition 2 can be modified to yield the following. Suppose that $r$ is a nonnegative function defined on $[0, \infty)$ and such that $\lim u^{-1} r(u) = 0$ as $u \to 0$. Then there are positive measures $M$ on $C$ which are not proper Lévy measures even though they satisfy the requirement that $\int r[\|x\|] M(dx) < \infty$.

In view of this, the following result gives the best sufficient condition in terms of integrability of functions of the norm.

**Theorem 1.** *Let $M$ be a positive Borel measure on a separable Banach space. If $\int \min[1, \|x\|]M(dx) < \infty$ then $M$ is the Lévy measure of an infinitely divisible distribution carried by the Banach space.*

**Proof.** Let $M_n$ be the restriction of $M$ to the set $R_n = \{x; (1/(n+1)) < \|x\| \leq (1/n)\}$. Then $\|M_n\|$ is finite.

Let $Z_n = \Sigma_{j \leq N_n} Y_{n,j}$ where all the variables $N_n$, $Y_{n,j}$, $j = 1, 2, \ldots$, are independent, $N_n$ has a Poisson distribution with $EN_n = \|M_n\|$ and the $Y_{n,j}$ have the distribution $M_n/\|M_n\|$. Then $\|Z_n\| \leq \Sigma_{j \leq N_n} \|Y_{n,j}\|$ and therefore

$$E\|Z_n\| \leq (EN_n)E\|Y_{n,j}\| \leq \|M_n\|n^{-1}.$$

If $\int \min[1, \|x\|]M(dx) < \infty$, the series $\Sigma Z_n$ converges in $L_1$-norm. Hence the restriction of $M$ to the unit ball is a proper Lévy measure. The part of $M$ situated outside the unit ball is finite. It is also a Lévy measure. Hence the result.

3. **A sufficient condition and some of its applications.** Conditions which imply that $M$ is a proper Lévy measure can be obtained through an inequality of Garsia, Rodemich and Rumsey [3].

Before stating them let us note the easily verified fact that $M$ is a proper Lévy measure if and only if the symmetrized measure $M_s$ defined by $M_s(A) = M(A) + M[(-A)]$ is likewise a proper Lévy measure. In view of this we shall concentrate on symmetric measures.

The following notation will be used. Let $\rho$ be a continuous increasing function defined on $[0, \infty)$. Assume that $\rho(0) = 0$ and let $\rho(u) = \rho(|u|)$ for $u < 0$. For every $f \in C[0, 1]$ let

$$\phi(s, t; f) = [f(s) - f(t)][\rho(s - t)]^{-1}$$

and let $B(f)$ be the integral

$$B(f) = \int_0^1 \int_0^1 \exp\{|\phi(s, t; f)|\} \, ds \, dt.$$

Finally, if $M$ is a positive measure on $C[0, 1]$ let $J(s, t; M)$ be the integral

$$J(s, t; M) = \int \{\exp[\phi(s, t; f)] - 1 - \phi(s, t; f)\}M(df).$$

**Theorem 2.** *Assume that the function $\rho$ is such that*

(i) $\int(\rho(u)/u)\, du < \infty$.

*For a fixed* $b < \infty$ *let* $\mathfrak{M}$ *be the set of all* $\sigma$*-finite positive symmetric measures such that*

(ii) $\int f^2(0)M(df) \leq b$,

(iii) $\int_0^1 \int_0^1 \exp\{J(s, t; M)\}\, ds\, dt \leq b$,

*then each* $M \in \mathfrak{M}$ *is the Lévy measure of an infinitely divisible distribution* $P_M$ *on* $C = C[0, 1]$ *and the set* $\{P_M; M \in \mathfrak{M}\}$ *is tight on* $C$.

**Proof.** Consider first a finite $M \in \mathfrak{M}$. This is the Lévy measure of a distribution $P_M$ for which

$$\int \exp\{|\phi(s, t; f)|\} P_M(df) \leq 2 \int \exp\{\phi(s, t; f)\} P_M(df) \leq 2 \exp\{J(s, t; M)\}.$$

Take a $\beta > 0$ and let $S_\beta$ be the bounded equicontinuous set of all $f \in C$ such that $|f(0)|^2 \leq \beta$ and

$$|f(t) - f(s)| \leq 8 \int_0^{|t-s|} \log(\beta/u^2)\, d\rho(u),$$

for all $(s, t)$. According to [3] the inequalities $B(f) \leq \beta$ and $|f(0)|^2 \leq \beta$ imply that $f \in S_\beta$. The assumptions (ii) and (iii) and Markov's inequality yield immediately that $P_M(S_\beta) \geq 1 - 3(b/\beta)$. Hence the result for the finite $M \in \mathfrak{M}$. An arbitrary $M \in \mathfrak{M}$ is the limit of an increasing sequence $M_n$ of finite elements of $\mathfrak{M}$. The tightness obtained in the first part of the argument shows then that $P_{M_n}$ converges to a $P_M$ whose Lévy measure is $M$. This completes the proof of the theorem.

**Corollary.** *Let* $M$ *be a positive, symmetric measure on a compact subset* $K$ *of* $C[0, 1]$. *Suppose that there is an even continuous function* $q$ *defined on* $[-1, +1]$ *and increasing on* $[0, 1]$ *such that* $|f(s) - f(t)| \leq q^2(s - t)$ *for all pairs* $(s, t)$ *and all* $f \in K$.

*Assume that for some* $\alpha \in (0, 1)$ *one has*

(a) $\int \|f\|^{2-\alpha} M(df) < \infty$,

(b) $\int_0^1 [q(u)]^\alpha |\log(u/2)|^{-1/2} (du/u) < \infty$.

*Then* $\mathfrak{M}$ *is a proper Lévy measure.*

**Proof.** Let $c$ be a number such that $c^2 > 2e \int \|f\|^{2-\alpha} M(df)$. Define a function $\rho$ by $\rho(u) = c q^\alpha(u) |\log(u/2)|^{-1/2}$. It can be verified that condition (b) implies that such a $\rho$ satisfies the requirements imposed for Theorem 2. Define $\phi(s, t; f)$ as before and consider first the set $A$ of all $f \in K$ such that

$|\phi(s, t; f)| \leq 1$ for all pairs $(s, t)$. Let $M_1$ be the measure $M$ restricted to this set.

The integral $J(s, t; M_1)$ can be bounded by

$$J(s, t; M_1) \leq \frac{1}{2} e \int |\phi(s, t; f)|^2 M(df)$$

Also

$$|\phi(s, t; f)|^2 \leq 4\|f\|^{2-\alpha} |f(s) - f(t)|^\alpha \rho^{-2}(s - t)$$
$$\leq 4c^{-2}\|f\|^{2-\alpha}|f(s) - f(t)|^\alpha q^{-2\alpha}(s - t)|\log \tfrac{1}{2}|s - t||.$$

Therefore

$$J(s, t; M_1) \leq 4ec^{-2}\left[\int \|f\|^{2-\alpha} M(df)\right]|\log \tfrac{1}{2}|s - t||.$$

Since in this expression the coefficient of the logarithmic term is strictly less than unity we conclude that $\int_0^1 \int_0^1 \exp\{J(s, t; M_1)\}\, ds\, dt < \infty$. Hence $M_1$ is a proper Lévy measure by Theorem 2.

To complete the proof it is sufficient to show that $M(A^c) < \infty$, since $M$ restricted to $A^c$ will then be a proper Lévy measure.

For this purpose take a $k > 2$ and write $\Delta = |f(t) - f(s)|$ and $q = q(s - t)$ for short. Then

$$|c|f(t) - f(s)|\rho^{-1}(s - t)|^k = \Delta^{2-\alpha}\Delta^{k-(2-\alpha)}q^{-k\alpha}[\log \tfrac{1}{2}|s - t|]^{k/2}$$
$$\leq (2\|f\|)^{2-\alpha} q^{2[k-(2-\alpha)]-k\alpha}|\log \tfrac{1}{2}|s - t||^{k/2}.$$

The condition (b) implies that $q^\alpha(u)|\log (u/2)|^{1/2}$ tends to zero as $u \to 0$. Therefore if $k(1 - \alpha) > 2 - \alpha$, we can assert that

$$|c|f(t) - f(s)|\rho^{-1}(s - t)|^k \leq c_1\|f\|^{2-\alpha}$$

for all $(s, t)$ and a suitable constant $c_1$. The finiteness of $M(A^c)$ follows by Markov's inequality.

**Examples.** (1) Let $\rho_2(u)$ be the function defined by letting $r^2(s, t) = \int (f(s) - f(t))^2 M(df)$ and $\rho_2(u) = \sup\{r(s, t);\ |s - t| \leq u\}$. Suppose that $M = \sum_{n=1}^\infty \lambda_n(\delta_{\{f_n\}} + \delta_{\{-f_n\}})$ with $\lambda_n > c > 0$, $f_n \in C[0, 1]$. Then condition (iii) of Theorem 2 is satisfied with $\rho(u) = \rho_2(u)$.

(2) Using the same method as in the Corollary one can show that $M$ is a proper Lévy measure if there is a $\rho$ satisfying the conditions of Theorem 2 such that one has always $|f(s) - f(t)| \leq c_f(s, t)\rho(s - t)$ with

$$\int \sup\{c_f^2(s, t);\ |s - t| < u\} M(df) < \infty$$

for some $u \in (0, 1)$.

**Remarks.** (1) Theorem 2 can easily be extended to the space of continuous functions on a $k$-dimensional cube, or other compact metric spaces, using the results of Garsia [7] and Preston [8].

(2) When checking condition (iii) of Theorem 2 one can always remove a part $M'$ of $M$ such that $\int \min[1, \|x\|] M'(dx) < \infty$, since the sum of two proper Lévy measures is a proper Lévy measure.

(3) Our application of Garsia's formula relies on the function $\exp u$ instead of the $\exp u^2$ used for Gaussian variables, since introducing the square may easily make the expectations infinite.

(4) It is known (see LeCam [6]) that tightness of sets of infinitely divisible distributions may be used to prove central limit theorems. This will be investigated in another publication.

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