REMARK ON NILPOTENT ORBITS

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ABSTRACT. If $G$ is a reductive Lie group and $\Theta_f = \text{Ad}(G)^*f$ is a nilpotent coadjoint orbit with invariant real polarization $\mathfrak{p}$, then $\Theta_f$ is identified as an open $G$-orbit on the cotangent bundle of $G/P$.

Introduction. Let $R^{4,1}$ denote real 5-space with the bilinear form $b(x, y) = x^1 y^1 + \ldots + x^4 y^4 - x^5 y^5$ and let $C^+$ denote its forward light cone \{ $x \in R^{4,1}$: $b(x, x) = 0$ and $x^5 > 0$ \}. The rays in $C^+$ form a 3-sphere $S^3$, and so the identity component $SO(4, 1)$ of the orthogonal group of $R^{4,1}$ acts on the cotangent bundle $T^*(S^3)$. This observation is due to B. Kostant, who noted that $SO(4, 1)$ is transitive on the symplectic manifold $T^*(S^3)$—(zero section) and asked Y. Akyildiz to identify that space as a coadjoint orbit for $SO(4, 1)$. Akyildiz identified it as a nilpotent coadjoint orbit, and Kostant noted from dimension considerations that the nilpotent elements in question must be regular-nilpotent. Kostant and I then conjectured that if $G$ is semisimple, $P$ is a minimal parabolic subgroup, and $\mathfrak{e} \in \mathfrak{p}$ is a regular-nilpotent element of $\mathfrak{g}$, then $\text{Ad}(G) \cdot \mathfrak{e}$ is an open $G$-orbit on the cotangent bundle $T^*(G/P)$. Here note that $SO(4, 1)/(\text{minimal parabolic}) = SO(4)/SO(3) = S^3$. The conjecture is proved as Corollary 2 below.

We refer to [1] for the language of polarizations.

Lemma. Let $\mathfrak{g}$ be a real Lie algebra, $f \in \mathfrak{g}^*$ a linear functional on $\mathfrak{g}$, and $\mathfrak{q} \subset \mathfrak{g}_C$ a complex polarization for $f$. If $f(q) = 0$ then $q$ is real in the sense $q = \mathfrak{p}_C$, where $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$.

Proof. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $E^0$ and $D^0$ its respective analytic subgroups for $e = (q + \overline{q}) \cap \mathfrak{g}$ and $b = (q \cap \overline{q}) \cap \mathfrak{g}$.

$\text{Ad}(D^0)^* \cdot f$ is open in the affine subspace $f + e^\perp$ of $\mathfrak{g}^*$, and $f \in e^\perp$ because $f(q) = 0$, so also $\text{Ad}(E^0)^* \cdot f$ is open in $f + e^\perp$. As $\mathfrak{q}^f \subset b \subset e$, now dim $e =$.
Theorem. 2. Let $G$ be a Lie group, $f \in \mathfrak{g}^*$, $q$ a complex polarization for $f$ such that $f(q) = 0$, and $\mathfrak{p} = q \cap \mathfrak{g}$. Let $P$ be a closed subgroup of $G$ with Lie algebra $\mathfrak{p}$ such that $G^f \subseteq P$. Then $\tilde{\mathcal{O}}_f = \text{Ad}(G)^* \cdot f$ is equivariantly diffeomorphic to an open $G$-orbit in the cotangent bundle $\mathcal{T}^*(G/P)$.

Proof. As in the lemma, $\text{Ad}(P)^* \cdot f$ is open in the subspace $f + \mathfrak{p}^\perp = \mathfrak{p}^\perp$ of $\mathfrak{g}^*$.

$G/P$ has tangent space $\mathfrak{g}/\mathfrak{p}$, hence cotangent space $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp$, all this as $P$-modules. Thus $\mathcal{T}^*(G/P)$ is the $G$-homogeneous bundle $G \times_P \mathfrak{p}^\perp$.

It consists of all classes

$$\begin{align*}
[\mathfrak{g}, \mathfrak{y}] &= \{(gp, \text{Ad}(p)^*y) : p \in P \subseteq G \times \mathfrak{p}^\perp\}
\end{align*}$$

with quotient differentiable structure from $G \times \mathfrak{p}^\perp$ and with left action of $G$ given by $g'[\mathfrak{g}, \mathfrak{y}] = [g'g, \mathfrak{y}]$. Define a $G$-orbit on $\mathcal{T}^*(G/P)$ by

$$\Omega_f = G([1, f]) = \{[g, f] \in G \times_P \mathfrak{p}^\perp : g \in G\}.$$ 

Then

$$\begin{align*}
\dim \Omega_f &= \dim (G/P) + \dim (\text{Ad}(P)^* \cdot f) = \dim \mathfrak{g} - \dim \mathfrak{p} + \dim \mathfrak{p}^\perp \\
&= \dim (G \times_P \mathfrak{p}^\perp) = \dim \mathcal{T}^*(G/P),
\end{align*}$$

so $\Omega_f$ is open in $\mathcal{T}^*(G/P)$.

Map the orbit $\tilde{\mathcal{O}}_f$ to $\Omega_f$ by $\text{Ad}(g)^* / \mapsto [g, f]$. This is well defined, for if $\text{Ad}(g)^* / = \text{Ad}(g')^* /$ then $g' = gp$ with $p \in G^f \subseteq P$ so $[g', f] = [gp, f] = [g, \text{Ad}(p)^*] = [g, f]$. It is visibly $G$-equivariant with image $\Omega_f$, and is one-to-one because $[g, f] = [g', f]$ forces $[g^{-1}g', f] = [1, f]$ whence $g^{-1}g' \in G^f \subseteq P$. Q.E.D.

We now suppose that $G$ is a reductive Lie group, i.e. that its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the center and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. We also suppose that $\mathfrak{g}$ has a nondegenerate $\text{Ad}(G)$-invariant symmetric bilinear form $\langle , \rangle$. That is automatic for example if $\{\text{Ad}(g) : g \in G\}$ is pre-compact, e.g. when if $g \in G$ then $\text{Ad}(g)$ is an inner automorphism on $\mathfrak{g}_C$, in particular when $G$ is connected. The form $\langle , \rangle$ gives a $G$-equivariant isomorphism of $\mathfrak{g}$ to $\mathfrak{g}^*$, say $x \mapsto x^*$, by $x^*(y) = \langle x, y \rangle$. We say that $x$ and $x^*$,  

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2 Originally we started with Corollary 1 below (same proof). Alan Weinstein suggested the possibility of a more general formulation.
and their $G$-orbits, are "nilpotent" when $x \in [\mathfrak{g}, \mathfrak{g}]$ with $\text{ad}(x): \mathfrak{g} \to \mathfrak{g}$ nilpotent as linear transformation.

Combining [2, Theorem 2.2] and [3, Proposition 2.3.2] we have

**Lemma.** Let $G$ be reductive as above, $x \in \mathfrak{g}$, and $\mathfrak{q}$ a complex polarization for $x^*$. Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}_C$, and $x^*(\mathfrak{q}) = 0$ if and only if $x$ is nilpotent.

Now we can prove

**Corollary 1.** Let $G$ be reductive as above, $e \in \mathfrak{g}$ a nilpotent element, $\mathfrak{q}$ a complex polarization for $e^*$, and $P$ the parabolic subgroup of $G$ with Lie algebra $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$. Then $\text{Ad}(G) \cdot e$ is equivariantly diffeomorphic to an open $G$-orbit on $\mathcal{F}^*(G/P)$ if, and only if, the polarization $\mathfrak{q}$ is $\text{Ad}(G^e)$-invariant.

**Proof.** If $\mathfrak{q}$ is $\text{Ad}(G^e)$-invariant, then $G^e \subset P$, and the Theorem realizes $\text{Ad}(G) \cdot e$ as an open $G$-orbit on $\mathcal{F}^*(G/P)$. If $\text{Ad}(G) \cdot e$ is equivariantly diffeomorphic to an open $G$-orbit on $\mathcal{F}^*(G/P)$, then the diffeomorphism must be given as in the proof of the Theorem; that requires $G^e \subset P$, and so $\mathfrak{q}$ is $\text{Ad}(G^e)$-invariant. Q.E.D.

If $e \in \mathfrak{g}$ is regular-nilpotent then $e$ is contained in a unique minimal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Now $e$ is in the nilradical $\mathfrak{p}_n = \mathfrak{p}^\perp$, so $\mathfrak{q} = \mathfrak{p}_C$ is a complex polarization for $e^*$, and $\mathfrak{q}$ is $\text{Ad}(G^e)$-invariant by uniqueness of $\mathfrak{p}$. Thus Corollary 1 specializes to

**Corollary 2.** Let $G$ be reductive as above, $e \in \mathfrak{g}$ a regular-nilpotent element, and $P$ the unique minimal parabolic subgroup of $G$ whose Lie algebra contains $e$. Then $\text{Ad}(G) \cdot e$ is $G$-equivariantly diffeomorphic to an open $G$-orbit on $\mathcal{F}^*(G/P)$.

**Remarks.** 1. When $[\mathfrak{g}, \mathfrak{g}]$ is isomorphic to the Lie algebra of $SO(n, 1)$, then in Corollary 2 we have $\text{Ad}(P) \cdot e = \mathfrak{p}_n - 10 = \mathfrak{p}_n^\perp - \{0\}$, so the open $G$-orbit is $\mathcal{F}^*(G/P)$—(the zero cross section).

2. Let $P$ be any parabolic subgroup of $G$, $\mathfrak{p}$ its Lie algebra, and $\mathfrak{p}_n$ the nilradical of $\mathfrak{p}$. R. W. Richardson and C. C. Moore independently showed that there are open $\text{Ad}(P)$-orbits on $\mathfrak{p}_n$. If $\text{Ad}(P) \cdot e$ is one of them, then $e^*(\mathfrak{p}) = 0$, and a dimension count shows that $\mathfrak{q} = \mathfrak{p}_C$ is a complex polarization for $e^*$. Conversely if $e \in \mathfrak{g}$ is nilpotent and $\mathfrak{q}$ is a complex polarization
for $e^*$, then our lemmas show $q = p_C$ with $p$ parabolic in $g$, $e \in p^\perp = p_n$ and $\text{Ad}(P) \cdot e$ open in $p_n$. But there are many instances in which $q$ is not $\text{Ad}(G^e)$-invariant. The invariant case is characterized in Corollary 1.

REFERENCES


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