

## REMARK ON NILPOTENT ORBITS

JOSEPH A. WOLF<sup>1</sup>

**ABSTRACT.** If  $G$  is a reductive Lie group and  $\mathcal{O}_f = \text{Ad}(G)^* f$  is a nilpotent coadjoint orbit with invariant real polarization  $\mathfrak{p}$ , then  $\mathcal{O}_f$  is identified as an open  $G$ -orbit on the cotangent bundle of  $G/P$ .

**Introduction.** Let  $R^{4,1}$  denote real 5-space with the bilinear form  $b(x, y) = x^1 y^1 + \dots + x^4 y^4 - x^5 y^5$  and let  $\mathcal{C}^+$  denote its forward light cone  $\{x \in R^{4,1} : b(x, x) = 0 \text{ and } x^5 > 0\}$ . The rays in  $\mathcal{C}^+$  form a 3-sphere  $S^3$ , and so the identity component  $\text{SO}(4, 1)$  of the orthogonal group of  $R^{4,1}$  acts on the cotangent bundle  $\mathcal{T}^*(S^3)$ . This observation is due to B. Kostant, who noted that  $\text{SO}(4, 1)$  is transitive on the symplectic manifold  $\mathcal{T}^*(S^3)$ —(zero section) and asked Y. Akyildiz to identify that space as a coadjoint orbit for  $\text{SO}(4, 1)$ . Akyildiz identified it as a nilpotent coadjoint orbit, and Kostant noted from dimension considerations that the nilpotent elements in question must be regular-nilpotent. Kostant and I then conjectured that if  $G$  is semisimple,  $P$  is a minimal parabolic subgroup, and  $e \in \mathfrak{p}$  is a regular-nilpotent element of  $\mathfrak{g}$ , then  $\text{Ad}(G) \cdot e$  is an open  $G$ -orbit on the cotangent bundle  $\mathcal{T}^*(G/P)$ . Here note that  $\text{SO}(4, 1)/(\text{minimal parabolic}) = \text{SO}(4)/\text{SO}(3) = S^3$ . The conjecture is proved as Corollary 2 below.

We refer to [1] for the language of polarizations.

**Lemma.** Let  $\mathfrak{g}$  be a real Lie algebra,  $f \in \mathfrak{g}^*$  a linear functional on  $\mathfrak{g}$ , and  $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$  a complex polarization for  $f$ . If  $f(\mathfrak{q}) = 0$  then  $\mathfrak{q}$  is real in the sense  $\mathfrak{q} = \mathfrak{p}_{\mathbb{C}}$  where  $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$ .

**Proof.** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  and  $E^0$  and  $D^0$  its respective analytic subgroups for

$$\mathfrak{e} = (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g} \quad \text{and} \quad \mathfrak{b} = (\mathfrak{q} \cap \bar{\mathfrak{q}}) \cap \mathfrak{g}.$$

$\text{Ad}(D^0)^* \cdot f$  is open in the affine subspace  $f + e^{\perp}$  of  $\mathfrak{g}^*$ , and  $f \in e^{\perp}$  because  $f(\mathfrak{q}) = 0$ , so also  $\text{Ad}(E^0)^* \cdot f$  is open in  $f + e^{\perp}$ . As  $\mathfrak{g}^f \subset \mathfrak{b} \subset \mathfrak{e}$ , now  $\dim \mathfrak{e} =$

Received by the editors May 21, 1974.

AMS (MOS) subject classifications (1970). Primary 22E45, 53C99; Secondary 22E15, 22E43.

<sup>1</sup> Research partially supported by NSF Grant GP-16651.

$\dim(f + e^\perp) - \dim \mathfrak{g}^f = \dim \mathfrak{h}$ , this forces  $e = \mathfrak{h}$ , and we conclude  $q = \bar{q}$ . Q.E.D.

**Theorem.<sup>2</sup>** *Let  $G$  be a Lie group,  $f \in \mathfrak{g}^*$ ,  $q$  a complex polarization for  $f$  such that  $f(q) = 0$ , and  $\mathfrak{p} = q \cap \mathfrak{g}$ . Let  $P$  be a closed subgroup of  $G$  with Lie algebra  $\mathfrak{p}$  such that  $G^f \subset P$ . Then  $\mathcal{O}_f = \text{Ad}(G)^* \cdot f$  is equivariantly diffeomorphic to an open  $G$ -orbit in the cotangent bundle  $\mathcal{J}^*(G/P)$ .*

**Proof.** As in the lemma,  $\text{Ad}(P)^* \cdot f$  is open in the subspace  $f + \mathfrak{p}^\perp = \mathfrak{p}^\perp$  of  $\mathfrak{g}^*$ .

$G/P$  has tangent space  $\mathfrak{g}/\mathfrak{p}$ , hence cotangent space  $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp$ , all this as  $P$ -modules. Thus  $\mathcal{J}^*(G/P)$  is the  $G$ -homogeneous bundle  $G \times_P \mathfrak{p}^\perp$ . It consists of all classes

$$[g, y] = \{(gp^{-1}, \text{Ad}(p)^*y) : p \in P\} \subset G \times \mathfrak{p}^\perp$$

with quotient differentiable structure from  $G \times \mathfrak{p}^\perp$  and with left action of  $G$  given by  $g'[g, y] = [g'g, y]$ . Define a  $G$ -orbit on  $\mathcal{J}^*(G/P)$  by

$$\Omega_f = G([1, f]) = \{[g, f] \in G \times_P \mathfrak{p}^\perp : g \in G\}.$$

Then

$$\begin{aligned} \dim \Omega_f &= \dim(G/P) + \dim(\text{Ad}(P)^* \cdot f) = \dim \mathfrak{g} - \dim \mathfrak{p} + \dim \mathfrak{p}^\perp \\ &= \dim(G \times_P \mathfrak{p}^\perp) = \dim \mathcal{J}^*(G/P), \end{aligned}$$

so  $\Omega_f$  is open in  $\mathcal{J}^*(G/P)$ .

Map the orbit  $\mathcal{O}_f$  to  $\Omega_f$  by  $\text{Ad}(g)^*f \mapsto [g, f]$ . This is well defined, for if  $\text{Ad}(g)^*f = \text{Ad}(g')^*f$  then  $g' = gp$  with  $p \in G^f \subset P$  so  $[g', f] = [gp, f] = [g, \text{Ad}(p)^*f] = [g, f]$ . It is visibly  $G$ -equivariant with image  $\Omega_f$ , and is one-to-one because  $[g, f] = [g', f]$  forces  $[g^{-1}g', f] = [1, f]$  whence  $g^{-1}g' \in G^f \subset P$ . Q.E.D.

We now suppose that  $G$  is a reductive Lie group, i.e. that its Lie algebra  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{c}$  where  $\mathfrak{c}$  is the center and  $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$  is semisimple. We also suppose that  $\mathfrak{g}$  has a nondegenerate  $\text{Ad}(G)$ -invariant symmetric bilinear form  $\langle, \rangle$ . That is automatic for example if  $\{\text{Ad}(g)|_{\mathfrak{c}} : g \in G\}$  is precompact, e.g. when if  $g \in G$  then  $\text{Ad}(g)$  is an inner automorphism on  $\mathfrak{g}_{\mathfrak{c}}$ , in particular when  $G$  is connected. The form  $\langle, \rangle$  gives a  $G$ -equivariant isomorphism of  $\mathfrak{g}$  to  $\mathfrak{g}^*$ , say  $x \rightarrow x^*$ , by  $x^*(y) = \langle x, y \rangle$ . We say that  $x$  and  $x^*$ ,

---

<sup>2</sup>Originally we started with Corollary 1 below (same proof). Alan Weinstein license or copyright restrictions may apply to redistribution; see <https://www.ams.org/journal-terms-of-use> suggested the possibility of a more general formulation.

and their  $G$ -orbits, are "nilpotent" when  $x \in [\mathfrak{g}, \mathfrak{g}]$  with  $\text{ad}(x): \mathfrak{g} \rightarrow \mathfrak{g}$  nilpotent as linear transformation.

Combining [2, Theorem 2.2] and [3, Proposition 2.3.2] we have

**Lemma.** *Let  $G$  be reductive as above,  $x \in \mathfrak{g}$ , and  $\mathfrak{q}$  a complex polarization for  $x^*$ . Then  $\mathfrak{q}$  is a parabolic subalgebra of  $\mathfrak{g}_{\mathbb{C}}$ , and  $x^*(\mathfrak{q}) = 0$  if and only if  $x$  is nilpotent.*

Now we can prove

**Corollary 1.** *Let  $G$  be reductive as above,  $e \in \mathfrak{g}$  a nilpotent element,  $\mathfrak{q}$  a complex polarization for  $e^*$ , and  $P$  the parabolic subgroup of  $G$  with Lie algebra  $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$ . Then  $\text{Ad}(G) \cdot e$  is equivariantly diffeomorphic to an open  $G$ -orbit on  $\mathcal{J}^*(G/P)$  if, and only if, the polarization  $\mathfrak{q}$  is  $\text{Ad}(G^e)$ -invariant.*

**Proof.** If  $\mathfrak{q}$  is  $\text{Ad}(G^e)$ -invariant, then  $G^e \subset P$ , and the Theorem realizes  $\text{Ad}(G) \cdot e$  as an open  $G$ -orbit on  $\mathcal{J}^*(G/P)$ . If  $\text{Ad}(G) \cdot e$  is equivariantly diffeomorphic to an open  $G$ -orbit on  $\mathcal{J}^*(G/P)$ , then the diffeomorphism must be given as in the proof of the Theorem; that requires  $G^e \subset P$ , and so  $\mathfrak{q}$  is  $\text{Ad}(G^e)$ -invariant. Q.E.D.

If  $e \in \mathfrak{g}$  is regular-nilpotent then  $e$  is contained in a unique minimal parabolic subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$ . Now  $e$  is in the nilradical  $\mathfrak{p}_n = \mathfrak{p}^\perp$ , so  $\mathfrak{q} = \mathfrak{p}_{\mathbb{C}}$  is a complex polarization for  $e^*$ , and  $\mathfrak{q}$  is  $\text{Ad}(G^e)$ -invariant by uniqueness of  $\mathfrak{p}$ . Thus Corollary 1 specializes to

**Corollary 2.** *Let  $G$  be reductive as above,  $e \in \mathfrak{g}$  a regular-nilpotent element, and  $P$  the unique minimal parabolic subgroup of  $G$  whose Lie algebra contains  $e$ . Then  $\text{Ad}(G) \cdot e$  is  $G$ -equivariantly diffeomorphic to an open  $G$ -orbit on  $\mathcal{J}^*(G/P)$ .*

**Remarks.** 1. When  $[\mathfrak{g}, \mathfrak{g}]$  is isomorphic to the Lie algebra of  $\text{SO}(n, 1)$ , then in Corollary 2 we have  $\text{Ad}(P) \cdot e = \mathfrak{p}_n - \{0\} = \mathfrak{p}^\perp - \{0\}$ , so the open  $G$ -orbit is  $\mathcal{J}^*(G/P)$ —(the zero cross section).

2. Let  $P$  be any parabolic subgroup of  $G$ ,  $\mathfrak{p}$  its Lie algebra, and  $\mathfrak{p}_n$  the nilradical of  $\mathfrak{p}$ . R. W. Richardson and C. C. Moore independently showed that there are open  $\text{Ad}(P)$ -orbits on  $\mathfrak{p}_n$ . If  $\text{Ad}(P) \cdot e$  is one of them, then<sup>3</sup>  $e^*(\mathfrak{p}) = 0$ , and a dimension count shows that  $\mathfrak{q} = \mathfrak{p}_{\mathbb{C}}$  is a complex polarization for  $e^*$ . Conversely if  $e \in \mathfrak{g}$  is nilpotent and  $\mathfrak{q}$  is a complex polarization

for  $e^*$ , then our lemmas show  $\mathfrak{q} = \mathfrak{p}_C$  with  $\mathfrak{p}$  parabolic in  $\mathfrak{g}$ ,  $e \in \mathfrak{p}^\perp = \mathfrak{p}_n$  and  $\text{Ad}(P) \cdot e$  open in  $\mathfrak{p}_n$ . But there are many instances in which  $\mathfrak{q}$  is not  $\text{Ad}(G^e)$ -invariant. The invariant case is characterized in Corollary 1.

## REFERENCES

1. L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups*, *Invent. Math.* **14** (1971), 255–354. MR 45 #2092.
2. H. Ozeki and M. Wakimoto, *On polarizations of certain homogeneous spaces*, *Proc. Japan Acad.* **48** (1972), 1–4. MR 47 #402.
3. L. P. Rothschild and J. A. Wolf, *Representations of semisimple groups associated to nilpotent orbits*, *Ann. Sci. École Norm. Sup.* **7** (1974), 155–174.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY,  
CALIFORNIA 94720

*Current address:* Institute of Mathematics, The Hebrew University, Jerusalem,  
Israel