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REMARK ON NILPOTENT ORBITS

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ABSTRACT. If G is a reductive Lie group and $\mathcal{O}_f = \mathrm{Ad}(G)^* f$ is a nilpotent coadjoint orbit with invariant real polarization $\mathfrak{p}$, then $\mathcal{O}_f$ is identified as an open G-orbit on the cotangent bundle of G/P.

Introduction. Let $R^{4,1}$ denote real 5-space with the bilinear form $b(x, y) = x^1y^1 + \ldots + x^4y^4 - x^5y^5$ and let $\mathcal{C}^+$ denote its forward light cone $\{x \in R^{4,1}: b(x, x) = 0 \text{ and } x^5 > 0\}$. The rays in $\mathcal{C}^+$ form a 3-sphere S^3, and so the identity component SO(4, 1) of the orthogonal group of $R^{4,1}$ acts on the cotangent bundle $\mathcal{T}^*(S^3)$. This observation is due to B. Kostant, who noted that SO(4, 1) is transitive on the symplectic manifold $\mathcal{T}^*(S^3)$–(zero section) and asked Y. Akyildiz to identify that space as a coadjoint orbit for SO(4, 1). Akyildiz identified it as a nilpotent coadjoint orbit, and Kostant noted from dimension considerations that the nilpotent elements in question must be regular-nilpotent. Kostant and I then conjectured that if G is semisimple, P is a minimal parabolic subgroup, and $e \in \mathfrak{p}$ is a regular-nilpotent element of $\mathfrak{g}$, then $\mathrm{Ad}(G) \cdot e$ is an open G-orbit on the cotangent bundle $\mathcal{T}^*(G/P)$. Here note that SO(4, 1)/(minimal parabolic) = SO(4)/SO(3) = S^3. The conjecture is proved as Corollary 2 below.

We refer to [1] for the language of polarizations.

Lemma. *Let $\mathfrak{g}$ be a real Lie algebra, $f \in \mathfrak{g}^*$ a linear functional on $\mathfrak{g}$, and $\mathfrak{q} \subset \mathfrak{g}_C$ a complex polarization for f. If $f(\mathfrak{q}) = 0$ then $\mathfrak{q}$ is real in the sense $\mathfrak{q} = \mathfrak{p}_C$ where $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$.*

Proof. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and E^0 and D^0 its respective analytic subgroups for

$$\mathfrak{e} = (\mathfrak{q} + \overline{\mathfrak{q}}) \cap \mathfrak{g} \quad \text{and} \quad \mathfrak{d} = (\mathfrak{q} \cap \overline{\mathfrak{q}}) \cap \mathfrak{g}.$$

$\mathrm{Ad}(D^0)^* \cdot f$ is open in the affine subspace $f + \mathfrak{e}^\perp$ of $\mathfrak{g}^*$, and $f \in \mathfrak{e}^\perp$ because $f(\mathfrak{q}) = 0$, so also $\mathrm{Ad}(E^0)^* \cdot f$ is open in $f + \mathfrak{e}^\perp$. As $\mathfrak{g}^f \subset \mathfrak{d} \subset \mathfrak{e}$, now dim $\mathfrak{e}$ =

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$\dim(f + \mathfrak{e}^\perp) - \dim \mathfrak{g}^f = \dim \mathfrak{b}$, this forces $\mathfrak{e} = \mathfrak{b}$, and we conclude $\mathfrak{q} = \overline{\mathfrak{q}}$. Q.E.D.

Theorem.[2] *Let G be a Lie group, $f \in \mathfrak{g}^*$, $\mathfrak{q}$ a complex polarization for f such that $f(\mathfrak{q}) = 0$, and $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$. Let P be a closed subgroup of G with Lie algebra $\mathfrak{p}$ such that $G^f \subset P$. Then $\mathcal{O}_f = \mathrm{Ad}(G)^* \cdot f$ is equivariantly diffeomorphic to an open G-orbit in the cotangent bundle $\mathcal{T}^*(G/P)$.*

Proof. As in the lemma, $\mathrm{Ad}(P)^* \cdot f$ is open in the subspace $f + \mathfrak{p}^\perp = \mathfrak{p}^\perp$ of $\mathfrak{g}^*$.

G/P has tangent space $\mathfrak{g}/\mathfrak{p}$, hence cotangent space $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp$, all this as P-modules. Thus $\mathcal{T}^*(G/P)$ is the G-homogeneous bundle $G \times_P \mathfrak{p}^\perp$. It consists of all classes

$$[g, y] = \{(gp^{-1}, \mathrm{Ad}(p)^* y) : p \in P\} \subset G \times \mathfrak{p}^\perp$$

with quotient differentiable structure from $G \times \mathfrak{p}^\perp$ and with left action of G given by $g'[g, y] = [g'g, y]$. Define a G-orbit on $\mathcal{T}^*(G/P)$ by

$$\Omega_f = G([1, f]) = \{[g, f] \in G \times_P \mathfrak{p}^\perp : g \in G\}.$$

Then

$$\dim \Omega_f = \dim(G/P) + \dim(\mathrm{Ad}(P)^* \cdot f) = \dim \mathfrak{g} - \dim \mathfrak{p} + \dim \mathfrak{p}^\perp$$

$$= \dim(G \times_P \mathfrak{p}^\perp) = \dim \mathcal{T}^*(G/P),$$

so Ω_f is open in $\mathcal{T}^*(G/P)$.

Map the orbit $\mathcal{O}_f$ to Ω_f by $\mathrm{Ad}(g)^* f \mapsto [g, f]$. This is well defined, for if $\mathrm{Ad}(g)^* f = \mathrm{Ad}(g')^* f$ then $g' = gp$ with $p \in G^f \subset P$ so $[g', f] = [gp, f] = [g, \mathrm{Ad}(p)^* f] = [g, f]$. It is visibly G-equivariant with image Ω_f, and is one-to-one because $[g, f] = [g', f]$ forces $[g^{-1}g', f] = [1, f]$ whence $g^{-1}g' \in G^f \subset P$. Q.E.D.

We now suppose that G is a reductive Lie group, i.e. that its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the center and $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. We also suppose that $\mathfrak{g}$ has a nondegenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form $\langle\, , \rangle$. That is automatic for example if $\{\mathrm{Ad}(g)|_{\mathfrak{c}} : g \in G\}$ is precompact, e.g. when if $g \in G$ then $\mathrm{Ad}(g)$ is an inner automorphism on $\mathfrak{g}_C$, in particular when G is connected. The form $\langle\, , \rangle$ gives a G-equivariant isomorphism of $\mathfrak{g}$ to $\mathfrak{g}^*$, say $x \longrightarrow x^*$, by $x^*(y) = \langle x, y\rangle$. We say that x and x^*,

[2] Originally we started with Corollary 1 below (same proof). Alan Weinstein suggested the possibility of a more general formulation.

and their G-orbits, are "nilpotent" when $x \in [\mathfrak{g}, \mathfrak{g}]$ with $\mathrm{ad}(x)\colon \mathfrak{g} \to \mathfrak{g}$ nilpotent as linear transformation.

Combining [2, Theorem 2.2] and [3, Proposition 2.3.2] we have

Lemma. *Let G be reductive as above, $x \in \mathfrak{g}$, and $\mathfrak{q}$ a complex polarization for x^*. Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}_C$, and $x^*(\mathfrak{q}) = 0$ if and only if x is nilpotent.*

Now we can prove

Corollary 1. *Let G be reductive as above, $e \in \mathfrak{g}$ a nilpotent element, $\mathfrak{q}$ a complex polarization for e^*, and P the parabolic subgroup of G with Lie algebra $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{g}$. Then $\mathrm{Ad}(G) \cdot e$ is equivariantly diffeomorphic to an open G-orbit on $\mathcal{T}^*(G/P)$ if, and only if, the polarization $\mathfrak{q}$ is $\mathrm{Ad}(G^e)$-invariant.*

Proof. If $\mathfrak{q}$ is $\mathrm{Ad}(G^e)$-invariant, then $G^e \subset P$, and the Theorem realizes $\mathrm{Ad}(G) \cdot e$ as an open G-orbit on $\mathcal{T}^*(G/P)$. If $\mathrm{Ad}(G) \cdot e$ is equivariantly diffeomorphic to an open G-orbit on $\mathcal{T}^*(G/P)$, then the diffeomorphism must be given as in the proof of the Theorem; that requires $G^e \subset P$, and so $\mathfrak{q}$ is $\mathrm{Ad}(G^e)$-invariant. Q.E.D.

If $e \in \mathfrak{g}$ is regular-nilpotent then e is contained in a unique minimal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Now e is in the nilradical $\mathfrak{p}_n = \mathfrak{p}^\perp$, so $\mathfrak{q} = \mathfrak{p}_C$ is a complex polarization for e^*, and $\mathfrak{q}$ is $\mathrm{Ad}(G^e)$-invariant by uniqueness of $\mathfrak{p}$. Thus Corollary 1 specializes to

Corollary 2. *Let G be reductive as above, $e \in \mathfrak{g}$ a regular-nilpotent element, and P the unique minimal parabolic subgroup of G whose Lie algebra contains e. Then $\mathrm{Ad}(G) \cdot e$ is G-equivariantly diffeomorphic to an open G-orbit on $\mathcal{T}^*(G/P)$.*

Remarks. 1. When $[\mathfrak{g}, \mathfrak{g}]$ is isomorphic to the Lie algebra of $\mathrm{SO}(n, 1)$, then in Corollary 2 we have $\mathrm{Ad}(P) \cdot e = \mathfrak{p}_n - \{0\} = \mathfrak{p}^\perp - \{0\}$, so the open G-orbit is $\mathcal{T}^*(G/P)$ –(the zero cross section).

2. Let P be any parabolic subgroup of G, $\mathfrak{p}$ its Lie algebra, and $\mathfrak{p}_n$ the nilradical of $\mathfrak{p}$. R. W. Richardson and C. C. Moore independently showed that there are open $\mathrm{Ad}(P)$-orbits on $\mathfrak{p}_n$. If $\mathrm{Ad}(P) \cdot e$ is one of them, then[3] $e^*(\mathfrak{p}) = 0$, and a dimension count shows that $\mathfrak{q} = \mathfrak{p}_C$ is a complex polarization for e^*. Conversely if $e \in \mathfrak{g}$ is nilpotent and $\mathfrak{q}$ is a complex polarization

[3] This was pointed out to me by B. Kostant.

for e^*, then our lemmas show $\mathfrak{q} = \mathfrak{p}_C$ with $\mathfrak{p}$ parabolic in $\mathfrak{g}$, $e \in \mathfrak{p}^\perp = \mathfrak{p}_n$ and $\mathrm{Ad}(P) \cdot e$ open in $\mathfrak{p}_n$. But there are many instances in which $\mathfrak{q}$ is not $\mathrm{Ad}(G^e)$-invariant. The invariant case is characterized in Corollary 1.

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