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HILBERT-SAMUEL FUNCTIONS OF COHEN-MACAULAY RINGS

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ABSTRACT. Let R be a local ring with a maximal ideal $\mathfrak{m}$. It is proved that in case R is a Cohen-Macaulay (C.M.) ring and $\dim \mathfrak{m}/\mathfrak{m}^2 - \dim R = 1$, then the multiplicity of R and its dimension determine uniquely the Hilbert-Samuel function of R. As a corollary we obtain that the C.M. property is determined by the Hilbert-Samuel function in case $\dim \mathfrak{m}/\mathfrak{m}^2 - \dim R = 1$. An example is given which shows that it is not so in case $\dim \mathfrak{m}/\mathfrak{m}^2 - \dim R > 1$.

In [1] Matlis has determined explicitly the Hilbert-Samuel functions of the one-dimensional Cohen-Macaulay rings (abbreviated C.M.) whose maximal ideal can be generated by two elements.

In our paper we shall extend this result to arbitrary n-dimensional C.M. rings with the property that $\dim \mathfrak{m}/\mathfrak{m}^2 \leq n + 1$. The possibility of such generalization was suggested to us by Professor Białynicki-Birula. Our method of proof is entirely different from that of Matlis. From our result we can deduce that the Hilbert-Samuel polynomial of an n-dimensional local ring determines the C.M. property in case $\dim \mathfrak{m}/\mathfrak{m}^2 = n + 1$.

In our paper R will always denote a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The Hilbert-Samuel function is defined as $H_R(n) = l(R/\mathfrak{m}^n)$ where $l(R/\mathfrak{m}^n)$ denotes the length of the (artinian) R-module $R/\mathfrak{m}^n$. Depth R is defined as a length of a maximal R-sequence contained in $\mathfrak{m}$. The ring R is said to be a C.M. ring if $\dim R =$ depth R.

Theorem 1. *Let R be a C.M. ring with the property that* $\dim \mathfrak{m}/\mathfrak{m}^2 - \dim R \leq 1$. *Then*

$$H_R(i) = \begin{cases} i & \text{if } i \leq l(R), \\ l(R) & \text{if } i > l(R) \end{cases} \qquad \text{in case } \dim R = 0.$$

If $\dim R = n > 0$ *then there exists a* (*local*) *C.M. ring* R' *with* $\dim R' = n - 1$

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and $\dim \mathfrak{m}'/\mathfrak{m}'^2 \leq n$ *where* $\mathfrak{m}'$ *is the maximal ideal of* R' *such that* $H_R(i) = \sum_{j=0}^{i-1} H_{R'}(j+1)$.

Remark 1. It follows from our theorem that the Hilbert-Samuel function of R is uniquely determined by the dimension and a number which is equal to the length of some artinian ring. It can be proved that this number is equal to $e(R)$, the multiplicity of R. So we obtain that in our case the Hilbert-Samuel function of R is uniquely determined by $\dim R$ and $e(R)$. Moreover $e(R) = 1$ if and only if R is a regular ring.

Remark 2. In case $\dim R = 1$ we obtain the existence of an artinian (local) ring R' of length equal to $e = e(R)$ whose maximal ideal is principal and such that $H_R(i) = \sum_{j=0}^{i-1} H_{R'}(j+1)$. So

$$H_R(i) = \begin{cases} i(i+1)/2 & \text{if } i \leq e, \\ ie - e(e-1)/2 & \text{if } i > e. \end{cases}$$

This proves the result of Matlis [1].

In case $\dim \mathfrak{m}/\mathfrak{m}^2 - \dim R = 0$ our result is obvious so we shall restrict ourselves to the case when $\dim \mathfrak{m}/\mathfrak{m}^2 - \dim R = 1$.

The above theorem is a consequence of the following

Proposition. *Let* R *be a complete ring with* $\dim R = n > 0$ *and with infinite residue field which moreover satisfies the assumptions of Theorem* 1. *Then there exists an element* $x \in \mathfrak{m}$ *such that* $(\mathfrak{m}^s : x) = \mathfrak{m}^{s-1}$ *for all* s, *where* $(\mathfrak{m}^s : x) = \{r \in R \mid rx \in \mathfrak{m}^s\}$.

The proof of the above Proposition will be preceded by a series of lemmas.

By Cohen's theorem a complete local ring R is a homomorphic image of some regular ring $\widetilde{R}$. We can assume that the kernel J of the homomorphism $\widetilde{R} \to R$ is contained in $\widetilde{\mathfrak{m}}^2$ where $\widetilde{\mathfrak{m}}$ denotes the maximal ideal in $\widetilde{R}$.

From now on we shall work under the assumptions of the Proposition.

Lemma 1. *The ideal* J *is generated by one element.*

Proof. Let $J = (f_1, f_2, \ldots, f_k)$ where $k > 1$ and let d denote the greatest common divisor of $\{f_i\} i = 1, 2, \ldots, k$ (R is a U.F.D.). If $d \in J$, then $J = (d)$. So we can suppose that $d \notin J$. Let $f_i = df_i'$ and $J' = (f_1', \ldots, f_k')$. We obtain that $J'd = 0$ in $\widetilde{R}/J$. So J' is contained in some prime ideal $\mathfrak{p}$ which is associated with J. We have $\dim \widetilde{R}/\mathfrak{p} = \dim \widetilde{R}/J = n$ because of the C.M. property of $\widetilde{R}/J$ [2]. On the other hand $\dim \widetilde{R} = \dim \widetilde{\mathfrak{m}}/\widetilde{\mathfrak{m}}^2 = \dim \mathfrak{m}/\mathfrak{m}^2 = n + 1$ ($J \subset \widetilde{\mathfrak{m}}^2$). The ideal J' is not contained in any minimal (principal) prime ideal of $\tilde{R}$, so $\dim \tilde{R}/J' < \dim \tilde{R} - 1 = n$ which contradicts the fact that $\dim \tilde{R}/\mathfrak{p} = n$ $(J' \subset \mathfrak{p})$. This accomplishes the proof of Lemma 1.

The ring $\mathrm{Gr}(R) = \amalg_{i=0}^{\infty} \mathrm{Gr}(R)_i$ will denote the graded ring associated with the $\mathfrak{m}$-adic filtration of R. It is well known [3] that, in case R is a regular ring with $\dim R = d$, the ring $\mathrm{Gr}(R)$ is isomorphic with $k[x_1, \ldots, x_d]$. From now on f will stand for a generator of J which exists by Lemma 1.

Lemma 2. *With our assumptions* $\mathrm{Gr}(R) = \mathrm{Gr}(\tilde{R}/(f)) \simeq k[x_1, \ldots, x_{n+1}]/J'$ *where* J' *is also generated by one element.*

The proof is routine and therefore we shall omit it. A generator of J' will be denoted by f^*.

Proof of the Proposition. Let $x \in \mathfrak{m}\backslash\mathfrak{m}^2$ and suppose that $(\mathfrak{m}^s : x)$ strictly contains $\mathfrak{m}^{s-1}$. Then it is easy to see that x is a zero divisor contained in $\mathrm{Gr}(R)_1$. It follows that in case there does not exist an element with the property claimed in the Proposition, each element in $\mathrm{Gr}(R)_1$ is a zero divisor. So $\mathrm{Gr}(R)_1 \subset \bigcup \mathfrak{p}_i$ where $\mathfrak{p}_i$ are the associated prime ideals of $\mathrm{Gr}(R)$. Because of the fact that $k = \mathrm{Gr}(R)_0$ is infinite, the 1-forms $\mathrm{Gr}(R)_1 \subset \mathfrak{p}_i$ for some i. It follows that the ideal generated by images of $x_1, \ldots, x_{n+1}$ in $\mathrm{Gr}(R)$ annihilates some nonzero y in $\mathrm{Gr}(R)$. So $x_i y \in J' = (f^*)$. We easily obtain that $y \in (f^*)$, which contradicts the fact that $y \neq 0$ in $\mathrm{Gr}(R)$.

Proof of Theorem 1. If $\dim R = 0$ then each power of $\mathfrak{m}$ is a principal ideal. So $H_R(i) = \Sigma_{j=0}^{i-1} l(\mathfrak{m}^j/\mathfrak{m}^{j+1}) = i$ if $i \leq l(R)$ and $H_R(i) = l(R)$ if $i \geq l(R)$. Let $\dim R = n > 0$. First we shall suppose that R is complete and has an infinite residue field. By the Proposition there exists $x \in \mathfrak{m}$ such that $(\mathfrak{m}^s : x) = \mathfrak{m}^{s-1}$ for all s. We consider the exact sequence of R-modules:

$$0 \longrightarrow \ker \varphi \longrightarrow R/\mathfrak{m}^j \xrightarrow{\ \varphi\ } R/\mathfrak{m}^{j+1} \longrightarrow R/Rx + \mathfrak{m}^{j+1} \longrightarrow 0$$

where φ denotes multiplication by x. By our choice of x, the ideal $\ker \varphi = 0$. We obtain that $l(\mathfrak{m}^j/\mathfrak{m}^{j+1}) = l(R/Rx + \mathfrak{m}^{j+1})$. So

$$H_R(i) = \sum_{j=0}^{i-1} l(\mathfrak{m}^j/\mathfrak{m}^{j+1}) = \sum_{j=0}^{i-1} l(R/Rx + \mathfrak{m}^{j+1})$$

$$= \sum_{j=0}^{i-1} H_{R'}(j+1) \quad \text{where } R' = R/Rx.$$

It follows easily from our choice of x that $x \in \mathfrak{m}\backslash\mathfrak{m}^2$ and is not a zero divisor. The ring R is a C.M. ring with $\dim R' = n - 1$ and $\dim \mathfrak{m}'/\mathfrak{m}'^2 = n$.

Now let R be an arbitrary ring satisfying the conditions of Theorem 1 with $\dim R = n > 0$. Following Rees we consider the ring $R[X]_{\mathfrak{m}[X]}$ which has an infinite residue field. The C.M. property is preserved and the Hilbert-Samuel function remains unchanged. It follows that all our assumptions of Theorem 1 are unaltered. Nothing will change if we take the completion of $R[X]_{\mathfrak{m}[X]}$. From the above considerations Theorem 1 follows immediately.

For any local ring P let W_P denote the Hilbert-Samuel polynomial determined by the Hilbert-Samuel function of P.

Theorem 2. *Let S be a C.M. ring such that* $\dim \mathfrak{m}_S/\mathfrak{m}_S^2 - \dim S \leq 1$. *If R is a local ring such that $W_R = W_S$ then R is a C.M. ring.*

Proof. As before we can assume that R has an infinite residue field.

We define inductively the sequence of local rings. We put $R_0 = R$. Suppose we have defined R_i. We distinguish two cases.

1°. If $\mathfrak{m}_i$, the maximal ideal of R_i consists entirely of zero divisors, we put $R_{i+1} = R_i/J$ where $J = \bigcup_{k=1}^{\infty}(0: \mathfrak{m}_i^k)$ and $(0: \mathfrak{m}_i^k) = \{x \in R_i | \mathfrak{m}_i^k x = 0\}$. The maximal ideal of R_{i+1} does not consist anymore entirely of zero divisors.

2°. If $\mathfrak{m}_i$ contains a nonzero divisor we put $R_{i+1} = R_i/Rx$ where x has the property that $(\mathfrak{m}_i^n : x) = \mathfrak{m}_{i-1}^{n-1}$ for almost all n. Such an element exists by [2, Chapter II, Corollary of Theorem 2].

In case 1° we shall show that

$$W_{R_i} = W_{R_{i+1}} + a \tag{1}$$

where a is a nonzero constant. In fact we have $l_{R_{i+1}}(j) \leq l_{R_i}(j)$ for all j with equality for almost all j. (The equality holds for $n > k$ where k has the property that $\mathfrak{m}^k \cap J = 0$; such a k exists because J is an artinian R-module.) It follows that the Hilbert-Samuel polynomials W_{R_i} and $W_{R_{i+1}}$ differ by a constant. We have $l_{R_{i+1}}(j) < l_{R_i}(j)$ if j has the property that $J \subset \mathfrak{m}_i^j$ and $J \not\subset \mathfrak{m}_i^{j+1}$. So the above-mentioned constant is nonzero.

In case 2° it follows from the proof of Theorem 1 that for large n, the function $H_{R_i}(n) = \sum_{j=0}^{n-1} H_{R_{i+1}}(j+1) + a$ where a is a constant. So for large n we obtain that

$$W_{R_i}(n) = \sum_{j=0}^{n-1} W_{R_{i+1}}(j+1) + b \tag{2}$$

where b is a constant which depends on a and the differences $W_{R_{i+1}}(j+1) - H_{R_{i+1}}(j+1)$ for small j.

In particular we obtain that $e(R) = e(R_i)$ for all i since the leading coefficient of W_{R_i} is equal to $e(R_i)$ divided by the factorial of $\dim R_i$.

After some number of steps of our inductive construction we obtain a zero-dimensional ring which is C.M. Let t be the smallest number with the property that R_t is C.M. The symbol D_i will denote the polynomial $W_{R_i} - W_{R_t[[X_1,\dots,X_s]]}$ where $s = \dim R_i - \dim R_t$ and $0 \le i < t$. We shall prove by induction on s that $D_i \neq 0$ and $\deg D_i = s$. If $s = 0$ then $i = t-1$ and by (1) $D_{t-1} = a \neq 0$ (the maximal ideal of R_{t-1} consists entirely of zero divisors).

Suppose our assertion is true for some s and let us take such an i that $\dim R_i - \dim R_t = s$ and $\dim R_{i-1} > \dim R_i$. Then for large n

$$W_{R_{i-1}}(n) = \sum_{j=0}^{n-1} W_{R_i}(j+1) + b \quad \text{by (2).}$$

For large n we have

$$\begin{aligned} D_{i-1}(n) &= W_{R_{i-1}}(n) - W_{R_t[[X_1,\dots,X_{s+1}]]}(n) \\ &= \sum_{j=0}^{n-1} W_{R_i}(j+1) + b - \sum_{j=0}^{n-1} W_{R_t[[X_1,\dots,X_s]]}(j+1) \\ &= \sum_{j=0}^{n-1} (W_{R_i} - W_{R_t[[X_1,\dots,X_s]]})(j+1) + b = \sum_{j=0}^{n-1} D_i(j+1) + b. \end{aligned}$$

So $\deg D_{i-1} = \deg D_i + 1 = s+1$ and $D_{i-1} \neq 0$. If $\dim R_{i-2} - \dim R_t = s+1$, then $\deg D_{i-2} = s+1$, since $W_{R_{i-1}}$ and $W_{R_{i-2}}$ differ by a constant. This finishes the proof of our assertion.

In particular for $i = 0$ we have $D_0 = W_R - W_{R_t[[X_1,\dots,X_d]]} \neq 0$ if $t > 0$ where $d = \dim R - \dim R_t$. Let us put $S = R_t[[X_1, \dots, X_d]]$. The ring S is a C.M. ring such that $e(S) = e(R)$, $\dim R = \dim S$, $\dim \mathfrak{m}_S/\mathfrak{m}_S^2 - \dim S \le 1$ and finally, $W_R \neq W_S$ if R is a non-C.M. ring. In view of Remark 1 the Theorem 2 is proved.

Example. Set

$$R = k[X, Y, Z, T]/(X^2, Y^3, XY^2, XT, YT, ZT, T^2)_{(X,Y,Z,T)}$$

and

$$S = k[X, Y, Z, T]/(Y^2, Z^2, XY, YZ, ZX, X^3)_{(X,Y,Z,T)}.$$

Both rings are one dimensional. The ring R is not a C.M. ring because t is annihilated by its maximal ideal, while S is a C.M. ring because t is not a zero divisor in it. It is easy to calculate that

$$H_R(i) = H_S(i) = \begin{cases} 1 & \text{if } i = 1, \\ 5i - 5 & \text{if } i > 1. \end{cases}$$

This shows that the assumption concerning the difference between the dimension of the tangent space and the dimension of the ring is essential.

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