HILBERT-SAMUEL FUNCTIONS OF
COHEN-MACAULAY RINGS

M. BORATYŃSKI AND J. ŚWIĘCICKA

ABSTRACT. Let $R$ be a local ring with a maximal ideal $m$. It is proved
that in case $R$ is a Cohen-Macaulay (CM.) ring and $\dim m/m^2 - \dim R = 1$, 
then the multiplicity of $R$ and its dimension determine uniquely the Hilbert-
Samuel function of $R$. As a corollary we obtain that the CM. property is de-
termined by the Hilbert-Samuel function in case $\dim m/m^2 - \dim R = 1$. An 
example is given which shows that it is not so in case $\dim m/m^2 - \dim R > 1$.

In [1] Matlis has determined explicitly the Hilbert-Samuel functions of
the one-dimensional Cohen-Macaulay rings (abbreviated CM.) whose maximal
ideal can be generated by two elements.

In our paper we shall extend this result to arbitrary $n$-dimensional CM.
rings with the property that $\dim m/m^2 \leq n + 1$. The possibility of such gen-
eralization was suggested to us by Professor Bialynicki-Birula. Our method
of proof is entirely different from that of Matlis. From our result we can de-
duce that the Hilbert-Samuel polynomial of an $n$-dimensional local ring deter-
mines the CM. property in case $\dim m/m^2 = n + 1$.

In our paper $R$ will always denote a commutative noetherian local ring
with the maximal ideal $m$ and residue field $k = R/m$. The Hilbert-Samuel
function is defined as $H_R(n) = l(R/m^n)$ where $l(R/m^n)$ denotes the length of
the (artinian) $R$-module $R/m^n$. Depth $R$ is defined as a length of a maximal
$R$-sequence contained in $m$. The ring $R$ is said to be a C.M. ring if $\dim R =
\text{depth } R$.

Theorem 1. Let $R$ be a C.M. ring with the property that $\dim m/m^2 -
\dim R \leq 1$. Then

$$H_R(i) = \begin{cases} i & \text{if } i \leq l(R), \\
l(R) & \text{if } i > l(R) \end{cases}$$

in case $\dim R = 0$.

If $\dim R = n > 0$ then there exists a (local) C.M. ring $R'$ with $\dim R' = n - 1.
and \( \dim \frac{m'}{m'^2} \leq n \) where \( m' \) is the maximal ideal of \( R' \) such that \( H_{R'}(i) = \sum_{j=0}^{i-1} H_R(j+1) \).

**Remark 1.** It follows from our theorem that the Hilbert-Samuel function of \( R \) is uniquely determined by the dimension and a number which is equal to the length of some artinian ring. It can be proved that this number is equal to \( e(R) \), the multiplicity of \( R \). So we obtain that in our case the Hilbert-Samuel function of \( R \) is uniquely determined by \( \dim R \) and \( e(R) \). Moreover \( e(R) = 1 \) if and only if \( R \) is a regular ring.

**Remark 2.** In case \( \dim R = 1 \) we obtain the existence of an artinian (local) ring \( R' \) of length equal to \( e = e(R) \) whose maximal ideal is principal and such that \( H_{R'}(i) = \sum_{j=0}^{i-1} H_{R'}(j+1) \). So

\[
H_{R'}(i) = \begin{cases} 
i(i+1)/2 & \text{if } i \leq e, \\ \nu-e(e-1)/2 & \text{if } i > e. 
\end{cases}
\]

This proves the result of Matlis [1].

In case \( \dim \frac{m}{m^2} = \dim R = 0 \) our result is obvious so we shall restrict ourselves to the case when \( \dim \frac{m}{m^2} = \dim R = 1 \).

The above theorem is a consequence of the following

**Proposition.** Let \( R \) be a complete ring with \( \dim R = n > 0 \) and with infinite residue field which moreover satisfies the assumptions of Theorem 1. Then there exists an element \( x \in m \) such that \( \langle m^s : x \rangle = m^{s-1} \) for all \( s \), where \( \langle m^s : x \rangle = \{ r \in R | rx \in m^s \} \).

The proof of the above Proposition will be preceded by a series of lemmas.

By Cohen's theorem a complete local ring \( R \) is a homomorphic image of some regular ring \( \tilde{R} \). We can assume that the kernel \( J \) of the homomorphism \( \tilde{R} \to R \) is contained in \( \tilde{m} \) where \( \tilde{m} \) denotes the maximal ideal in \( \tilde{R} \).

From now on we shall work under the assumptions of the Proposition.

**Lemma 1.** The ideal \( J \) is generated by one element.

**Proof.** Let \( J = \langle f_1, f_2, \ldots, f_k \rangle \) where \( k > 1 \) and let \( d \) denote the greatest common divisor of \( \{ f_i \} i = 1, 2, \ldots, k \) (\( R \) is a U.F.D.). If \( d \not\in J \), then \( J = (d) \). So we can suppose that \( d \not\in J \). Let \( f_i = df'_i \) and \( J' = \langle f'_1, \ldots, f'_k \rangle \). We obtain that \( J'd = 0 \) in \( \tilde{R}/J \). So \( J' \) is contained in some prime ideal \( \mathfrak{p} \) which is associated with \( J \). We have \( \dim \tilde{R}/\mathfrak{p} = \dim \tilde{R}/J = n \) because of the C.M. property of \( \tilde{R}/J \) [2]. On the other hand \( \dim \tilde{R} = \dim \tilde{m}/\tilde{m}^2 = \dim \frac{m}{m^2} = n + 1 \) (\( \tilde{m} \subset \tilde{m}^2 \)). The ideal \( J' \) is not contained in any minimal (principal)
prime ideal of $\mathcal{R}$, so $\dim \mathcal{R}/j' < \dim \mathcal{R} - 1 = n$ which contradicts the fact that $\dim \mathcal{R}/p = n$ ($j' \subset p$). This accomplishes the proof of Lemma 1.

The ring $\text{Gr}(R) = \Pi_{i=0}^{\infty} \text{Gr}(R)_i$ will denote the graded ring associated with the $m$-adic filtration of $R$. It is well known [3] that, in case $R$ is a regular ring with $\dim R = d$, the ring $\text{Gr}(R)$ is isomorphic with $k[x_1, \ldots, x_d]$. From now on $f$ will stand for a generator of $J$ which exists by Lemma 1.

**Lemma 2.** With our assumptions $\text{Gr}(R) = \text{Gr}(\mathcal{R}/(f)) \cong k[x_1, \ldots, x_{n+1}]/J'$ where $J'$ is also generated by one element.

The proof is routine and therefore we shall omit it. A generator of $J'$ will be denoted by $f^*$.

**Proof of the Proposition.** Let $x \in m \setminus m^2$ and suppose that $(m^s : x)$ strictly contains $m^{s-1}$. Then it is easy to see that $x$ is a zero divisor contained in $\text{Gr}(R)_1$. It follows that in case there does not exist an element with the property claimed in the Proposition, each element in $\text{Gr}(R)_1$ is a zero divisor. So $\text{Gr}(R)_1 \subseteq \bigcup \mathfrak{p}_i$ where $\mathfrak{p}_i$ are the associated prime ideals of $\text{Gr}(R)$. Because of the fact that $k = \text{Gr}(R)_0$ is infinite, the 1-forms $\text{Gr}(R)_1 \subset \mathfrak{p}_i$ for some $i$. It follows that the ideal generated by images of $x_1, \ldots, x_{n+1}$ in $\text{Gr}(R)$ annihilates some nonzero $y$ in $\text{Gr}(R)$. So $x_i y \in J' = (f^*)$. We easily obtain that $y \in (f^*)$, which contradicts the fact that $y \neq 0$ in $\text{Gr}(R)$.

**Proof of Theorem 1.** If $\dim R = 0$ then each power of $m$ is a principal ideal. So $H_R(i) = \sum_{j=0}^{i-1} l(m^j/m^{j+1}) = i$ if $i \leq l(R)$ and $H_R(i) = l(R)$ if $i \geq l(R)$. Let $\dim R = n > 0$. First we shall suppose that $R$ is complete and has an infinite residue field. By the Proposition there exists $x \in m$ such that $(m^s : x) = m^{s-1}$ for all $s$. We consider the exact sequence of $R$-modules:

$$0 \rightarrow \ker \varphi \rightarrow R/m^i \xrightarrow{\varphi} R/m^{i+1} \rightarrow R/Rx + m^{i+1} \rightarrow 0$$

where $\varphi$ denotes multiplication by $x$. By our choice of $x$, the ideal $\ker \varphi = 0$. We obtain that $l(m^j/m^{j+1}) = l(R/Rx + m^{i+1})$. So

$$H_R(i) = \sum_{j=0}^{i-1} l(m^j/m^{j+1}) = \sum_{j=0}^{i-1} l(R/Rx + m^{i+1})$$

$$= \sum_{j=0}^{i-1} H_R'(j + 1)$$

where $R' = R/Rx$.

It follows easily from our choice of $x$ that $x \in m \setminus m^2$ and is not a zero divisor. The ring $R$ is a C.M. ring with $\dim R' = n - 1$ and $\dim m'/m'^2 = n$.

Now let $R$ be an arbitrary ring satisfying the conditions of Theorem 1 with $\dim R = n > 0$. Following Rees we consider the ring $R[X]/m[X]$ which
has an infinite residue field. The C.M. property is preserved and the Hilbert-Samuel function remains unchanged. It follows that all our assumptions of Theorem 1 are unaltered. Nothing will change if we take the completion of $\mathbb{R}[X]_{m[X]}$. From the above considerations Theorem 1 follows immediately.

For any local ring $P$ let $W_P$ denote the Hilbert-Samuel polynomial determined by the Hilbert-Samuel function of $P$.

**Theorem 2.** Let $S$ be a C.M. ring such that $\dim \frac{m_S}{m_S^2} - \dim S \leq 1$. If $R$ is a local ring such that $W_R = W_S$ then $R$ is a C.M. ring.

**Proof.** As before we can assume that $R$ has an infinite residue field.

We define inductively the sequence of local rings. We put $R_0 = R$. Suppose we have defined $R_i$. We distinguish two cases.

1°. If $m_i$, the maximal ideal of $R_i$ consists entirely of zero divisors, we put $R_{i+1} = R_i/J$ where $J = \bigcup_{k=1}^{\infty} (0 : m_i^k)$ and $(0 : m_i^k) = \{x \in R_i | m_i^k x = 0\}$.

The maximal ideal of $R_{i+1}$ does not consist anymore entirely of zero divisors.

2°. If $m_i$ contains a nonzero divisor we put $R_{i+1} = R_i/Rx$ where $x$ has the property that $(m_i^n : x) = m_i^{n-1}$ for almost all $n$. Such an element exists by [2, Chapter II, Corollary of Theorem 2].

In case 1° we shall show that

$$W_{R_i} = W_{R_{i+1}} + a$$

where $a$ is a nonzero constant. In fact we have $l_{R_{i+1}}(j) \leq l_{R_i}(j)$ for all $j$ with equality for almost all $j$. (The equality holds for $n > k$ where $k$ has the property that $m_i^k \cap J = 0$; such a $k$ exists because $J$ is an artinian $R$-module.)

It follows that the Hilbert-Samuel polynomials $W_{R_i}$ and $W_{R_{i+1}}$ differ by a constant. We have $l_{R_{i+1}}(j) < l_{R_i}(j)$ if $j$ has the property that $J \subseteq m_i^j$ and $J \notin m_i^{j+1}$. So the above-mentioned constant is nonzero.

In case 2° it follows from the proof of Theorem 1 that for large $n$, the function $H_{R_i}(n) = \sum_{j=0}^{n-1} H_{R_{i+1}}(j + 1) + a$ where $a$ is a constant. So for large $n$ we obtain that

$$W_{R_i}(n) = \sum_{j=0}^{n-1} W_{R_{i+1}}(j + 1) + b$$

where $b$ is a constant which depends on $a$ and the differences $W_{R_{i+1}}(j + 1) - H_{R_{i+1}}(j + 1)$ for small $j$.

In particular we obtain that $e(R) = e(R_i)$ for all $i$ since the leading coefficient of $W_{R_i}$ is equal to $e(R_i)$ divided by the factorial of $\dim R_i$.

After some number of steps of our inductive construction we obtain a zero-dimensional ring which is C.M. Let $t$ be the smallest number with the
property that $R_t$ is C.M. The symbol $D_i$ will denote the polynomial $W_{R, i} - W_{R, t}[[X_1, ..., X_s]]$ where $s = \dim R_i - \dim R_t$ and $0 \leq i < t$. We shall prove by induction on $s$ that $D_i \neq 0$ and $\deg D_i = s$. If $s = 0$ then $i = t - 1$ and by (1) $D_{t - 1} = a \neq 0$ (the maximal ideal of $R_{t - 1}$ consists entirely of zero divisors).

Suppose our assertion is true for some $s$ and let us take such an $i$ that $\dim R_i - \dim R_t = s$ and $\dim R_{i - 1} > \dim R_i$. Then for large $n$

$$W_{R, i - 1}(n) = \sum_{j=0}^{n-1} W_{R, i}(j + 1) + b \quad \text{by (2)}.$$ 

For large $n$ we have

$$D_{i - 1}(n) = W_{R, i - 1}(n) - W_{R, t}[[X_1, ..., X_{s+1}]](n)$$

$$= \sum_{j=0}^{n-1} W_{R, i}(j + 1) + b - \sum_{j=0}^{n-1} W_{R, t}[[X_1, ..., X_s]](j + 1)$$

$$= \sum_{j=0}^{n-1} (W_{R, i} - W_{R, t}[[X_1, ..., X_s]])(j + 1) + b = \sum_{j=0}^{n-1} D_t(j + 1) + b.$$ 

So $\deg D_{i - 1} = \deg D_i + 1 = s + 1$ and $D_{i - 1} \neq 0$. If $\dim R_{i - 2} - \dim R_t = s + 1$, then $\deg D_{i - 2} = s + 1$, since $W_{R, i - 1}$ and $W_{R, i - 2}$ differ by a constant. This finishes the proof of our assertion.

In particular for $i = 0$ we have $D_0 = W_R - W_{R, t}[[X_1, ..., X_d]] \neq 0$ if $t > 0$ where $d = \dim R - \dim R_t$. Let us put $S = R_t[[X_1, ..., X_d]]$. The ring $S$ is a C.M. ring such that $e(S) = e(R)$, $\dim R = \dim S$, $m_s/m_s^2 - \dim S \leq 1$ and finally, $W_R \neq W_S$ if $R$ is a non-C.M. ring. In view of Remark 1 the Theorem 2 is proved.

Example. Set $R = k[X, Y, Z, T]/(X^2, Y^3, XY^2, XT, YT, ZT, T^2)(X, Y, Z, T)$ and


Both rings are one dimensional. The ring $R$ is not a C.M. ring because $t$ is annihilated by its maximal ideal, while $S$ is a C.M. ring because $t$ is not a zero divisor in it. It is easy to calculate that

$$H_R(i) = H_S(i) = \begin{cases} 1 & \text{if } i = 1, \\ 5i - 5 & \text{if } i > 1. \end{cases}$$
This shows that the assumption concerning the difference between the dimension of the tangent space and the dimension of the ring is essential.

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REFERENCES