ON ERGODIC SEQUENCES OF MEASURES

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ABSTRACT. Let $Z$ be the group of integers and $\tilde{Z}$ its Bohr compactification. A sequence of probability measures $\{\mu_n, n = 1, 2, \ldots\}$ defined on $Z$ is said to be ergodic provided $\mu_n$ converges weakly to $\mu$, the Haar measure on $\tilde{Z}$. Let $A_n \subset Z$, $n = 1, 2, \ldots$, and define $\mu_n$ by $\mu_n(B) = |A_n \cap B|/|A_n|$ where $|B|$ is the cardinality of $B$. Then it is easy to show that if $|A_n \cap A_n + k|/|A_n| \to 1$ for every $k \in Z$, then $\mu_n$ is ergodic. Let $0 \leq p_k \leq 1$. In this paper we construct (random) sequences $\{\mu_n\}$ which are ergodic, and such that $\lim(|A_n \cap A_n + k|/|A_n|) = p_k$, for every $k \in Z$.

1. Introduction. Let $G$ be a locally compact abelian (l.c.a.) group. Let $\{\mu_n, n = 1, 2, \ldots\}$ be a sequence of probability measures defined on the Borel sets of $G$. We shall say that such a sequence is ergodic provided $\mu_n$ converges weakly to $\mu$, where $\mu$ is Haar measure on the Bohr compactification of $G$. The reason for this terminology is that ergodicity of such a sequence is necessary and sufficient for the generalized mean ergodic theorem to hold: let $\{U_g, g \in G\}$ be any strongly continuous unitary representation $G$ on a Hilbert space $H$. We say that the generalized mean ergodic theorem holds with respect to the sequence $\{\mu_n\}$ provided $\lim_{n \to \infty} \int_G U_g/\mu_n (dg) = Pf$ strongly, for every $f \in H$, where $P$ is the projection of $H$ on the space $\{f | U_g f = f, g \in G\}$.

As mentioned above and shown in [1], the generalized mean ergodic theorem holds for a sequence $\{\mu_n\}$ if and only if the sequence is ergodic.

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only if $G$ is $\sigma$-compact. (See, e.g., [2].) That the above condition is not necessary was shown in [1].

Now let $G = Z$, the group of integers, and let $\{k_j, j = 1, 2, \ldots\}$ be a sequence of positive integers. Let $A_n = \{k_1, \ldots, k_n\}$ and define $\mu_n$ as above by $\mu_n(B) = |A_n \cap B|/|A_n|$, where $|A|$ is the cardinality of $A$. Clearly each $\mu_n$ may be thought of as a measure on $\bar{Z}$, the Bohr compactification of $Z$, and it follows from the Levy continuity theorem that $\mu_n$ is ergodic if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i k_j \alpha} = 0 \quad \text{for } 0 < \alpha < 1. \quad (1)$$

From the criterion mentioned above it follows that such sequences will have this property provided $|A_n \cap A_n + k|/|A_n| \to 1$ for every integer $k$, where $A_n + k$ is $A_n$ translated by $k$. In a personal communication to one of the authors, Niederreiter [3] proved that given $\rho$, with $0 \leq \rho \leq 1$, and a positive integer $k$, there exists (in fact, he constructed it) a sequence $\{k_j\}$ such that

$$\lim_{n \to \infty} \frac{|A_n \cap A_n + k|}{|A_n|} = \rho,$$

and such that the corresponding measures $\{\mu_n\}$ are ergodic. In this note we prove that given $\rho_k$ with $0 \leq \rho_k \leq 1$, there exist (uncountably many) random sequences $\{k_j\}$ such that

(a) the corresponding sequences of measures are ergodic, and
(b) for every integer $k \neq 0$ we have

$$\lim_{n \to \infty} \frac{|A_n \cap A_n + k|}{|A_n|} = \rho_k.$$

In fact we show that (a) and (b) are true on a set of probability one.

As mentioned above, what must be shown is that (1) holds for all $\alpha$.

For the kind of sequences we construct it was shown by Robbins [4] that this holds for each $\alpha$ on a set of probability one, and the problem is to construct a single set of probability one such that the limiting relation holds for all $\alpha$ simultaneously. We also consider this problem for the group of reals.

2. The main result. Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with characteristic function

$$\phi(\alpha) = E e^{i\alpha X_1}.$$ 

Let $S_n = X_1 + \cdots + X_n$ and

$$T_n(\alpha) = \frac{1}{n} \sum_{k=1}^{n} e^{i\alpha S_k}.$$
Theorem. If \( \phi(\alpha) \neq 1 \) for \( \beta \leq \alpha \leq \gamma \) and if \( E(X_1) \) is finite, then \( \sup_{\beta \leq \alpha \leq \gamma} |T_n(\alpha)| \to 0 \) almost surely as \( n \to \infty \).

Proof. It is shown in the Lemma below that \( \sup_{\beta \leq \alpha \leq \gamma} E|T_n(\alpha)|^4 = O(n^{-2}) \). Now let \( k \) be integer valued and set

\[
A_n = \left[ \max_{\beta \leq k/n^{9/7} \leq \gamma} |T_n(k/n^{9/7})| \leq 1/n^{1/7} \right].
\]

Then, using Boole's and Chebychev's inequalities and the Lemma,

\[
P(A_n^c) \leq \sum_{k: \beta \leq k/n^{9/7} \leq \gamma} P\left( \left| T_n\left(\frac{k}{n^{9/7}}\right) \right| ^2 \geq \frac{1}{n^{2/7}} \right)
\]

\[
= \sum_{k: \beta \leq k/n^{9/7} \leq \gamma} O(n^{-10/7}) = O(n^{-1/7}).
\]

Also

\[
\sup_{\alpha} \left| \frac{d}{d\alpha} T_n(\alpha) \right| \leq \frac{1}{n} \sum_{k=1}^{n} |S_k|
\]

and, letting \( \mu = E|X_j| \),

\[
E|S_k| \leq \sum_{j=1}^{k} E|X_j| = k\mu.
\]

Let

\[
B_n = \left[ \sup_{\alpha} \left| \frac{d}{d\alpha} T_n(\alpha) \right| \leq n^{8/7} \right].
\]

By Markov's inequality

\[
P(B_n^c) \leq \frac{1}{n^{8/7}} E\left( \frac{1}{n} \sum_{k=1}^{n} |S_k| \right) \leq \frac{O(n^2)}{n^{15/7}} = O(n^{-1/7}).
\]

On \( A_n B_n \) we have

\[
\sup_{\beta \leq \alpha \leq \gamma} |T_n(\alpha)| \leq \frac{1}{n^{1/7}} + \frac{n^{8/7}}{2n^{9/7}} = O(n^{-1/7}),
\]

while \( P(A_n B_n^c) = O(n^{-1/7}) \) from the above estimates.

The proof is completed by using the Borel strong law of large numbers argument: for the subsequence \( \{n^{8}\} \),

\[
\sum_{n} P(A_n^c B_n^c) = \sum O(n^{-8/7}) < \infty
\]

and so \( \sup_{\beta \leq \alpha \leq \gamma} |T_{n,8}(\alpha)| \to 0 \) almost surely.
Now for any $m$, there exists an $n$ such that $n^8 \leq m < (n+1)^8$, and
\[
|(|T_m(\alpha)| - |T_n(\alpha)|)| \leq |T_m(\alpha) - T_n(\alpha)|
\]
\[
= \left| \frac{1}{m} \sum_{k=n^8+1}^{m} e^{iaS_k} - \left( \frac{1}{n^8} - \frac{1}{m} \right) \sum_{k=1}^{n^8} e^{iaS_k} \right|
\]
\[
\leq 2 \frac{m-n^8}{m} = O\left( \frac{1}{m^{1/8}} \right)
\]
uniformly in $\alpha$. The Theorem follows.

In the Lemma below we have occasion to use the relation valid for any complex $a$ and $b$ with $|a| \leq 1$ and $a \neq 1$:

\[
(*) \quad \left| \sum_{j=1}^{\nu} a^j b \right| = \left| \frac{a - a^{\nu+1}}{1 - a} b \right| \leq \frac{2|b|}{|1 - a|} = \frac{2|b|}{|1 - \bar{a}|}.
\]

We also use without further comment the fact that, since the $X_k$'s are independent and identically distributed, for any $k > j$,
\[
E e^{ia(S_k-S_j)} = \phi(\alpha)^{k-j}
\]
and the fact that $\phi(-\alpha) = \overline{\phi(\alpha)}$ and $|\phi(\alpha)| \leq 1$.

**Lemma.** If $\phi(\alpha) \neq 1$ for $\beta \leq \alpha \leq \gamma$ then
\[
\sup_{\beta \leq \alpha \leq \gamma} \frac{E|T_n(\alpha)|^4}{O(1/n^2)} = \frac{1}{n^2}.
\]

**Proof.** First observe that
\[
E|T_n(\alpha)|^4 = E\left( T_n(\alpha)^2 \right)^2 = \frac{1}{n^4} \sum_{j,k,l,m=1}^{n} E e^{ia(S_j+S_k-S_l-S_m)}
\]
\[
= \frac{1}{n^4} \left( \sum_{\nu} + \sum_{2} + \sum_{3} + \sum_{4} \right)
\]
where
\[
\sum_{\nu}(\alpha) = \sum_{j,k,l,m=1; \{j,k,l,m\}=\nu}^{n} E e^{ia(S_j+S_k-S_l-S_m)}
\]
The modulus of each term in the sum for $\sum_{\nu}$ is at most one, so $|\sum_{1}| = O(n)$ and $|\sum_{2}| = O(n^2)$. It is best to break $\sum_{3}$ into two sums, $\sum_{3}'$ and $\sum_{3}''$, where $\sum_{3}'$ consists of those terms of $\sum_{3}$ which for $j = k$ or $l = m$ and $\sum_{3}''$ consists of those
terms of \(\Sigma_3\) for which \(|j, k| \cap |l, m| = 1\). Then

\[
|\Sigma_3'(\alpha)| = 2 \left| \text{Re} \left( \sum_{j \neq k \neq l} E e^{i\alpha(2S_j - S_k - S_l)} \right) \right|
\]

\[
= 4 \left| \text{Re} \left( \sum_{j > k > l} \left( E e^{i\alpha(2S_j - S_k - S_l)} + E e^{i\alpha(2S_k - S_j - S_l)} + E e^{i\alpha(2S_l - S_j - S_k)} \right) \right) \right|
\]

\[
\leq \frac{8}{|1 - \phi(a)|} \left( \sum_{j > k} \phi(2\alpha)^{i-j-k} + \sum_{j > k} \overline{\phi(a)}^{i-j-k} + \sum_{k > l} \phi(2\alpha)^{k-l} \right)
\]

\[
= O(n^2)/|1 - \phi(a)|.
\]

where the inequality uses (*).

Similarly, using (*) in the last equality,

\[
|\Sigma_3''(\alpha)| = 4 \left| \text{Re} \left( \sum_{j > k} \phi(\alpha)^{i-j-k} \right) \right| = O(n^2)/|1 - \phi(a)|.
\]

To estimate \(\Sigma_4\) we will first write it in terms of ordered summation indices, \(j > k > l > m\). There are then six types of terms according to the position of the two positive signs among \(\pm S_j \pm S_k \pm S_l \pm S_m\). These can be coalesced into three types of terms by adding conjugates, to give

\[
|\Sigma_4(\alpha)| = 8 \left| \text{Re} \left( \sum_{j > k > l > m} \left( E e^{i\alpha(S_j + S_k - S_l - S_m)} + E e^{i\alpha(S_j - S_k + S_l - S_m)} \right) \right) \right|
\]

\[
= 8 \left| \text{Re} \left( \sum_{j > k > l > m} \left( \phi(\alpha)^{i-j-k} \phi(2\alpha)^{k-l} \phi(\alpha)^{l-m} + \phi(\alpha)^{i-j-k+l-m} \right) \right) \right|
\]

\[
\leq \frac{16}{|1 - \phi(a)|} \left( \sum_{k > l > m} \phi(2\alpha)^{k-l} \phi(\alpha)^{l-m} \right) + 2n \left| \sum_{l > m} \phi(\alpha)^{l-m} \right|
\]

\[
= O(n^2)/|1 - \phi(a)|^2.
\]

Here (*) is used twice, at the inequality and the last equality.

Combining these estimates,
\[
\operatorname{sup}_{\beta \leq \gamma} \frac{E|T_n(\alpha)|^4}{n^4} = \frac{O(n^2)}{n^4} \operatorname{sup}_{\beta \leq \gamma} \frac{1}{|1 - \phi(\alpha)|^2}.
\]

The hypothesis of the Lemma, together with the continuity of \( \phi \), imply that the supremum is finite, and the assertion is proved.

Let \( \mathcal{L}(d) \), for \( d > 0 \), denote the lattice \( \{0, \pm d, \pm 2d, \ldots\} \). We will say that \( X \) is an \( \mathcal{L}(d) \) lattice variable if \( P[X \in \mathcal{L}(d)] = 1 \) but there is no \( d' > d \) such that \( P[X \in \mathcal{L}(d')] = 1 \). It is well known that \( X \) is an \( \mathcal{L}(d) \) lattice variable if and only if \( \phi(\alpha) \neq 1 \) for \( 0 < \alpha < 2\pi/d \) and \( \phi(2\pi/d) = 1 \).

Corollary 1. If \( X_1 \) is an \( \mathcal{L}(1) \) lattice variable, then there exists a null set \( N \) not depending on \( \alpha \) such that, except on \( N \), \( T_n(\alpha) \rightarrow 0 \) as \( n \rightarrow \infty \) for every \( \alpha \neq 0 \) (mod \( 2\pi \)).

Proof. In this case the \( T_n(\alpha) \) as well as \( \phi(\alpha) \) are periodic of period \( 2\pi \). By the Theorem, for any \( k > 0 \), \( \operatorname{sup}_{1/k \leq \alpha \leq 2\pi k - 1/k} \left| T_n(\alpha) \right| \rightarrow 0 \) as \( n \rightarrow \infty \) except on a null set \( N_k \).

Evidently we can take \( N = \bigcup_{k=1}^\infty N_k \) to be the set specified in the corollary.

We will say that \( X \) is a nonlattice variable if \( X \) is not an \( \mathcal{L}(d) \) lattice variable for any \( d > 0 \) and if \( P[X \neq 0] > 0 \). This is the case if and only if \( \phi(\alpha) \neq 1 \) for any \( \alpha \neq 0 \). Then in the same way as for the first corollary we have the following result.

Corollary 2. If \( X_1 \) is a nonlattice variable, then there exists a null set \( N \) not depending on \( \alpha \) such that, except on \( N \), \( T_n(\alpha) \rightarrow 0 \) as \( n \rightarrow \infty \) for every \( \alpha \neq 0 \).

Let
\[
r_n(\delta) = \frac{||S_1, \ldots, S_n|| - \delta}{||S_1, \ldots, S_n||},
\]
We will consider only the case that \( X_1 > 0 \) so \( ||S_1, \ldots, S_n|| = n \).

Lattice case. Let \( X_1 \) be an \( \mathcal{L}(1) \) lattice variable. Then
\[
r_n(1) = \frac{1}{n} \sum_{k=2}^n X[X_k = 1] \xrightarrow{a.s.} P[X_1 = 1],
\]
as \( n \rightarrow \infty \).

Continuous case. Let \( X_1 \) be a nonlattice variable. Then
\[
r_n(\delta) = \frac{1}{n} \sum_{k=2}^n X[X_k = \delta \text{ or } X_k + X_{k+1} = \delta \text{ or } \ldots \text{ or } X_k + \ldots + X_n = \delta]
\]
\[
= \frac{1}{n} \sum_{j=0}^{n-2} \sum_{k=2}^{n-j} X_{A_{k,j}}
\]
where

\[ A_{k,j} = [X_k + \cdots + X_{k+j} = \delta]. \]

Let \( A_k = \bigcup_{j=0}^{\infty} A_{k,j} \) and let \( p_j = P(A_{1,j}) \) and \( p = P(A_1) \). Then by the ergodic theorem

\[
\frac{1}{n} \sum_{k=1}^{n} X_{A_{k,j}} \overset{a.s.}{\longrightarrow} p_j, \quad \frac{1}{n} \sum_{k=1}^{n} X_{A_k} \overset{a.s.}{\longrightarrow} p
\]

as \( n \to \infty \). But we have \( r_n(\delta) \leq n^{-1} \sum_{k=2}^{n} X_{A_k} \) so \( \lim \sup r_n(\delta) \leq p \). On the other hand, by Fatou's lemma,

\[
\liminf r_n(\delta) \geq \sum_{j=0}^{\infty} \liminf \left( \frac{1}{n} \sum_{k=2}^{n-j} X_{A_{k,j}} \right) = \sum_{j=0}^{\infty} p_j = p
\]

since \( A_1 = \bigcup_j A_{1,j} \) and the sets in the union are disjoint. Thus \( r_n(\delta) \overset{a.s.}{\longrightarrow} p \), as \( n \to \infty \). Clearly \( p = 0 \) except for at most a countable set of \( \delta \) values.

3. Concluding remarks. Now let \( X_1, X_2, \ldots \) be \( \mathcal{L}(1) \) random variables. Then it follows from the results in §2 that if we define \( k_j = X_1 + \cdots + X_j \) the corresponding sequences of measures satisfy conditions (a) and (b) of §1 with probability one.

The results also apply to the case when \( G = \mathbb{R} \), the additive group of real numbers. For if \( X_1, X_2, \ldots \) are positive nonlattice random variables we can apply Corollary 2 to show that \( \lim_n T_n(\alpha) = 0 \) for all \( \alpha \neq 0 \) with probability one. Thus if \( k_j = X_1 + \cdots + X_j \), and if \( Y(t), t \geq 0 \) is a stationary stochastic process we see that the mean ergodic theorem applies to the averages \( n^{-1} \sum_{j=1}^{n} Y(k_j) \) provided the process is second order.

REFERENCES


