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ON ERGODIC SEQUENCES OF MEASURES

J. R. BLUM[1] AND R. COGBURN

ABSTRACT. Let Z be the group of integers and $\bar{Z}$ its Bohr compactification. A sequence of probability measures $\{\mu_n, n = 1, 2, \ldots\}$ defined on Z is said to be ergodic provided μ_n converges weakly to $\bar{\mu}$, the Haar measure on $\bar{Z}$. Let $A_n \subset Z$, $n = 1, 2, \ldots$, and define μ_n by $\mu_n(B) = |A_n \cap B|/|A_n|$ where $|B|$ is the cardinality of B. Then it is easy to show that if $|A_n \cap A_n + k|/|A_n| \to 1$ for every $k \in Z$, then μ_n is ergodic. Let $0 \le p_k \le 1$. In this paper we construct (random) sequences $\{\mu_n\}$ which are ergodic, and such that $\lim(|A_n \cap A_n + k|/|A_n|) = p_k$, for every $k \in Z$.

1. Introduction. Let G be a locally compact abelian (l.c.a.) group. Let $\{\mu_n, n = 1, 2, \ldots\}$ be a sequence of probability measures defined on the Borel sets of G. We shall say that such a sequence is ergodic provided μ_n converges weakly to $\bar{\mu}$, where $\bar{\mu}$ is Haar measure on the Bohr compactification of G. The reason for this terminology is that ergodicity of such a sequence is necessary and sufficient for the generalized mean ergodic theorem to hold: let $\{U_g, g \in G\}$ be any strongly continuous unitary representation G on a Hilbert space H. We say that the generalized mean ergodic theorem holds with respect to the sequence $\{\mu_n\}$ provided $\lim_{n\to\infty} \int_G U_g f \mu_n(dg) = Pf$ strongly, for every $f \in H$, where P is the projection of H on the space $\{f \mid U_g f = f, g \in G\}$.

As mentioned above and shown in [1], the generalized mean ergodic theorem holds for a sequence $\{\mu_n\}$ if and only if the sequence is ergodic. Let $\{A_n\}$ be a sequence of Borel subsets of G, with $\mu(A_n) < \infty$, where μ is Haar measure on G. For $g \in G$ let $A_n g$ be A_n translated by g. Define the probability measures μ_n by

$$\mu_n(B) = \mu(A_n \cap B)/\mu(A_n),$$

for B a Borel set.

It is easy to show that if $\lim_{n\to\infty}(\mu(A_n \cap A_n g)/\mu(A_n)) = 1$ for every $g \in G$ then μ_n is an ergodic sequence. Such sequences will exist if and

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only if G is σ-compact. (See, e.g., [2].) That the above condition is not necessary was shown in [1].

Now let $G = Z$, the group of integers, and let $\{k_j, j = 1, 2, \ldots\}$ be a sequence of positive integers. Let $A_n = \{k_1, \ldots, k_n\}$ and define μ_n as above by $\mu_n(B) = |A_n \cap B|/|A_n|$, when $|A|$ is the cardinality of A. Clearly each μ_n may be thought of as a measure on $\overline{Z}$, the Bohr compactification of Z, and it follows from the Levy continuity theorem that μ_n is ergodic if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i k_j \alpha} = 0 \quad \text{for } 0 < \alpha < 1. \tag{1}$$

From the criterion mentioned above it follows that such sequences will have this property provided $|A_n \cap A_n + k|/|A_n| \to 1$ for every integer k, where $A_n + k$ is A_n translated by k. In a personal communication to one of the authors, Niederreiter [3] proved that given p, with $0 \le p \le 1$, and a positive integer k, there exists (in fact, he constructed it) a sequence $\{k_j\}$ such that

$$\lim_{n\to\infty} \frac{|A_n \cap A_n + k|}{|A_n|} = p,$$

and such that the corresponding measures $\{\mu_n\}$ are ergodic. In this note we prove that given p_k with $0 \le p_k \le 1$, there exist (uncountably many) random sequences $\{k_j\}$ such that

(a) the corresponding sequences of measures are ergodic, and

(b) for every integer $k \neq 0$ we have

$$\lim_{n\to\infty} \frac{|A_n \cap A_n + k|}{|A_n|} = p_k.$$

In fact we show that (a) and (b) are true on a set of probability one.

As mentioned above, what must be shown is that (1) holds for all α. For the kind of sequences we construct it was shown by Robbins [4] that this holds for *each* α on a set of probability one, and the problem is to construct a single set of probability one such that the limiting relation holds for all α simultaneously. We also consider this problem for the group of reals.

2. The main result. Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with characteristic function

$$\phi(\alpha) = Ee^{i\alpha X_1}.$$

Let $S_n = X_1 + \cdots + X_n$ and

$$T_n(\alpha) = \frac{1}{n} \sum_{k=1}^{n} e^{i\alpha S_k}.$$

Theorem. *If* $\phi(\alpha) \neq 1$ *for* $\beta \leq \alpha \leq \gamma$ *and if* $E(X_1)$ *is finite, then* $\sup_{\beta \leq \alpha \leq \gamma} |T_n(\alpha)| \to 0$ *almost surely as* $n \to \infty$.

Proof. It is shown in the Lemma below that $\sup_{\beta \leq \alpha \leq \gamma} E|T_n(\alpha)|^4 = O(n^{-2})$. Now let k be integer valued and set

$$A_n = \left[\max_{\beta \leq k/n^{9/7} \leq \gamma} |T_n(k/n^{9/7})| \leq 1/n^{1/7} \right].$$

Then, using Boole's and Chebychev's inequalities and the Lemma,

$$P(A_n^c) \leq \sum_{k:\ \beta \leq k/n^{9/7} \leq \gamma} P[|T_n(k/n^{9/7})|^2 \geq 1/n^{2/7}]$$

$$= \sum_{k:\ \beta \leq k/n^{9/7} \leq \gamma} O(n^{-10/7}) = O(n^{-1/7}).$$

Also

$$\sup_\alpha \left| \frac{d}{d\alpha} T_n(\alpha) \right| \leq \frac{1}{n} \sum_{k=1}^{n} |S_k|$$

and, letting $\mu = E|X_j|$,

$$E|S_k| \leq \sum_{j=1}^{k} E|X_j| = k\mu.$$

Let

$$B_n = \left[\sup_\alpha \left| \frac{d}{d\alpha} T_n(\alpha) \right| \leq n^{8/7} \right].$$

By Markov's inequality

$$P(B_n^c) \leq \frac{1}{n^{8/7}} E\left(\frac{1}{n} \sum_{k=1}^{n} |S_k| \right) \leq \frac{O(n^2)}{n^{15/7}} = O(n^{-1/7}).$$

On $A_n B_n$ we have

$$\sup_{\beta \leq \alpha \leq \gamma} |T_n(\alpha)| \leq \frac{1}{n^{1/7}} + \frac{n^{8/7}}{2n^{9/7}} = O(n^{-1/7}),$$

while $P(A_n B_n)^c = O(n^{-1/7})$ from the above estimates.

The proof is completed by using the Borel strong law of large numbers argument: for the subsequence $\{n^8\}$,

$$\sum_n P(A_{n^8} B_{n^8})^c = \sum O(n^{-8/7}) < \infty$$

and so $\sup_{\beta \leq \alpha \leq \gamma} |T_{n^8}(\alpha)| \to 0$ almost surely.

Now for any m, there exists an n such that $n^8 \le m < (n+1)^8$, and

$$\begin{aligned} |(|T_m(\alpha)| - |T_n(\alpha)|)| &\le |T_m(\alpha) - T_n(\alpha)| \\ &= \left| \frac{1}{m} \sum_{k=n^8+1}^{m} e^{i\alpha S_k} - \left(\frac{1}{n^8} - \frac{1}{m}\right) \sum_{k=1}^{n^8} e^{i\alpha S_k} \right| \\ &\le 2\,\frac{m-n^8}{m} = O\left(\frac{1}{m^{1/8}}\right) \end{aligned}$$

uniformly in α. The Theorem follows.

In the Lemma below we have occasion to use the relation valid for any complex a and b with $|a| \le 1$ and $a \ne 1$:

$$(*) \qquad \left| \sum_{j=1}^{\nu} a^j b \right| = \left| \frac{a - a^{\nu+1}}{1-a} b \right| \le \frac{2|b|}{|1-a|} = \frac{2|b|}{|1-\bar{a}|}.$$

We also use without further comment the fact that, since the X_k's are independent and identically distributed, for any $k > j$,

$$E e^{i\alpha(S_k - S_j)} = \phi(\alpha)^{k-j}$$

and the fact that $\phi(-\alpha) = \overline{\phi(\alpha)}$ and $|\phi(\alpha)| \le 1$.

Lemma. *If* $\phi(\alpha) \ne 1$ *for* $\beta \le \alpha \le \gamma$ *then*

$$\sup_{\beta \le \alpha \le \gamma} E|T_n(\alpha)|^4 = O(1/n^2).$$

Proof. First observe that

$$\begin{aligned} E|T_n(\alpha)|^4 &= E(T_n(\alpha)^2\, \overline{T_n(\alpha)^2}) \\ &= \frac{1}{n^4} \sum_{j,k,l,m=1}^{n} E e^{i\alpha(S_j + S_k - S_l - S_m)} \\ &= \frac{1}{n^4}\left(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4\right) \end{aligned}$$

where

$$\Sigma_\nu(\alpha) = \sum_{j,k,l,m=1;\, |\{j,k,l,m\}| = \nu}^{n} E e^{i\alpha(S_j + S_k - S_l - S_m)}.$$

The modulus of each term in the sum for Σ_ν is at most one, so $|\Sigma_1| = O(n)$ and $|\Sigma_2| = O(n^2)$.

It is best to break Σ_3 into two sums, Σ_3' and Σ_3'', where Σ_3' consists of those terms of Σ_3 which for $j = k$ or $l = m$ and Σ_3'' consists of those terms of Σ_3 for which $|\{j, k\} \cap \{l, m\}| = 1$. Then

$$\begin{aligned}\left|\Sigma_3'(\alpha)\right| &= 2\left|\operatorname{Re}\left(\sum_{j\neq k\neq l} Ee^{i\alpha(2S_j - S_k - S_l)}\right)\right| \\ &= 4\left|\operatorname{Re}\left(\sum_{j>k>l} \{Ee^{i\alpha(2S_j - S_k - S_l)} + Ee^{i\alpha(2S_k - S_j - S_l)} + Ee^{i\alpha(2S_l - S_j - S_k)}\}\right)\right| \\ &= 4\left|\operatorname{Re}\left(\sum_{j>k>l} \{\phi(2\alpha)^{j-k}\phi(\alpha)^{k-l} + \overline{\phi(\alpha)}^{j-k}\phi(\alpha)^{k-l} + \overline{\phi(\alpha)}^{j-k}\overline{\phi(2\alpha)}^{k-l}\}\right)\right| \\ &\le \frac{8}{|1-\phi(\alpha)|}\left(\left|\sum_{j>k} \phi(2\alpha)^{j-k}\right| + \left|\sum_{j>k} \overline{\phi(\alpha)}^{j-k}\right| + \left|\sum_{k>l} \overline{\phi(2\alpha)}^{k-l}\right|\right) \\ &= O(n^2)/|1-\phi(\alpha)|.\end{aligned}$$

where the inequality uses (*).

Similarly, using (*) in the last equality,

$$\begin{aligned}\left|\Sigma_3''(\alpha)\right| &= 4\left|\sum_{j\neq k\neq l} Ee^{i\alpha(S_j - S_k)}\right| \\ &= 8(n-2)\left|\operatorname{Re}\left(\sum_{j>k} \phi(\alpha)^{j-k}\right)\right| = O(n^2)/|1-\phi(\alpha)|.\end{aligned}$$

To estimate Σ_4 we will first write it in terms of ordered summation indices, $j > k > l > m$. There are then six types of terms according to the position of the two positive signs among $\pm S_j \pm S_k \pm S_l \pm S_m$. These can be coalesced into three types of terms by adding conjugates, to give

$$\begin{aligned}\left|\Sigma_4(\alpha)\right| &= 8\left|\operatorname{Re}\left(\sum_{j>k>l>m} \{Ee^{i\alpha(S_j + S_k - S_l - S_m)} + Ee^{i\alpha(S_j - S_k + S_l - S_m)}\right.\right. \\ &\qquad \left.\left. + Ee^{i\alpha(S_j - S_k - S_l + S_m)}\}\right)\right| \\ &= 8\left|\operatorname{Re}\left(\sum_{j>k>l>m} \{\phi(\alpha)^{j-k}\phi(2\alpha)^{k-l}\phi(\alpha)^{l-m} + \phi(\alpha)^{j-k+l-m}\right.\right. \\ &\qquad \left.\left. + \phi(\alpha)^{j-k}\overline{\phi(\alpha)}^{l-m}\}\right)\right| \\ &\le \frac{16}{|(1-\phi(\alpha)|}\left\{\left|\sum_{k>l>m} \phi(2\alpha)^{k-l}\phi(\alpha)^{l-m}\right| + 2n\left|\sum_{l>m} \phi(\alpha)^{l-m}\right|\right\} \\ &= O(n^2)/|1-\phi(\alpha)|^2.\end{aligned}$$

Here (*) is used twice, at the inequality and the last equality.

Combining these estimates,

$$\sup_{\beta \le \alpha \le \gamma} E|T_n(\alpha)|^4 = \frac{O(n^2)}{n^4} \sup_{\beta \le \alpha \le \gamma} \frac{1}{|1 - \phi(\alpha)|^2}.$$

The hypothesis of the Lemma, together with the continuity of ϕ, imply that the supremum is finite, and the assertion is proved.

Let $\mathfrak{L}(d)$, for $d > 0$, denote the lattice $\{0, \pm d, \pm 2d, \ldots\}$. We will say that X *is an* $\mathfrak{L}(d)$ *lattice variable* if $P[X \in \mathfrak{L}(d)] = 1$ but there is no $d' > d$ such that $P[X \in \mathfrak{L}(d')] = 1$. It is well known that X is an $\mathfrak{L}(d)$ lattice variable if and only if $\phi(\alpha) \neq 1$ for $0 < \alpha < 2\pi/d$ and $\phi(2\pi/d) = 1$.

Corollary 1. *If* X_1 *is an* $\mathfrak{L}(1)$ *lattice variable, then there exists a null set* N *not depending on* α *such that, except on* N, $T_n(\alpha) \to 0$ *as* $n \to \infty$ *for every* $\alpha \neq 0$ (mod 2π).

Proof. In this case the $T_n(\alpha)$ as well as $\phi(\alpha)$ are periodic of period 2π. By the Theorem, for any $k > 0$, $\sup_{1/k \le \alpha \le 2\pi - 1/k} |T_n(\alpha)| \to 0$ as $n \to \infty$ except on a null set N_k.

Evidently we can take $N = \bigcup_{k=1}^{\infty} N_k$ to be the set specified in the corollary.

We will say that X is a *nonlattice variable* if X is not an $\mathfrak{L}(d)$ lattice variable for any $d > 0$ and if $P[X \neq 0] > 0$. This is the case if and only if $\phi(\alpha) \neq 1$ for any $\alpha \neq 0$. Then in the same way as for the first corollary we have the following result.

Corollary 2. *If* X_1 *is a nonlattice variable, then there exists a null set* N *not depending on* α *such that, except on* N, $T_n(\alpha) \to 0$ *as* $n \to \infty$ *for every* $\alpha \neq 0$.

Let

$$r_n(\delta) = \frac{|\{S_1, \ldots, S_n\} \cap \{S_1 + \delta, \ldots, S_n + \delta\}|}{|\{S_1, \ldots, S_n\}|}.$$

We will consider only the case that $X_1 > 0$ so $|\{S_1, \ldots, S_n\}| = n$.

Lattice case. Let X_1 be an $\mathfrak{L}(1)$ lattice variable. Then

$$r_n(1) = \frac{1}{n} \sum_{k=2}^{n} \chi[X_k = 1] \xrightarrow{\text{a.s.}} P[X_1 = 1],$$

as $n \to \infty$.

Continuous case. Let X_1 be a nonlattice variable. Then

$$r_n(\delta) = \frac{1}{n} \sum_{k=2}^{n} \chi[X_k = \delta \text{ or } X_k + X_{k+1} = \delta \text{ or } \cdots \text{ or } X_k + \cdots + X_n = \delta]$$

$$= \frac{1}{n} \sum_{j=0}^{n-2} \sum_{k=2}^{n-j} \chi_{A_{k,j}}$$

where

$$A_{k,j} = [X_k + \cdots + X_{k+j} = \delta].$$

Let $A_k = \bigcup_{j=0}^{\infty} A_{k,j}$ and let $p_j = P(A_{1,j})$ and $p = P(A_1)$. Then by the ergodic theorem

$$\frac{1}{n}\sum_{k=1}^{n} \chi_{A_{k,j}} \xrightarrow{\text{a.s.}} p_j, \quad \frac{1}{n}\sum_{k=1}^{n} \chi_{A_k} \xrightarrow{\text{a.s.}} p$$

as $n \to \infty$. But we have $r_n(\delta) \leq n^{-1}\sum_{k=2}^{n} \chi_{A_k}$ so $\limsup r_n(\delta) \leq p$. On the other hand, by Fatou's lemma,

$$\liminf r_n(\delta) \geq \sum_{j=0}^{\infty} \liminf \left(\frac{1}{n}\sum_{k=2}^{n-j} \chi_{A_{k,j}}\right) = \sum_{j=0}^{\infty} p_j = p$$

since $A_1 = \bigcup_j A_{1,j}$ and the sets in the union are disjoint. Thus $r_n(\delta) \xrightarrow{\text{a.s.}} p$, as $n \to \infty$. Clearly $p = 0$ except for at most a countable set of δ values.

3. **Concluding remarks.** Now let $X_1, X_2, \ldots$ be $\mathfrak{L}(1)$ random variables. Then it follows from the results in §2 that if we define $k_j = X_1 + \cdots + X_j$ the corresponding sequences of measures satisfy conditions (a) and (b) of §1 with probability one.

The results also apply to the case when $G = R$, the additive group of real numbers. For if $X_1, X_2, \ldots$ are positive nonlattice random variables we can apply Corollary 2 to show that $\lim_n T_n(\alpha) = 0$ for all $\alpha \neq 0$ with probability one. Thus if $k_j = X_1 + \cdots + X_j$, and if $Y(t)$, $t \geq 0$ is a stationary stochastic process we see that the mean ergodic theorem applies to the averages $n^{-1}\sum_{j=1}^{n} Y(k_j)$ provided the process is second order.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87106 (Current address of R. Cogburn)

Current address (J. R. Blum): Department of Mathematics, University of Wisconsin–Milwaukee, Milwaukee, Wisconsin 53201