REFERENCES


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A CHARACTERIZATION OF THE KERNEL OF A CLOSED SET

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ABSTRACT. Let $S$ be a closed subset of some linear topological space such that $\text{int} \ker S \neq \emptyset$ and $\ker S \neq S$. Let $\mathcal{C}$ denote the collection of all maximal convex subsets of $S$ and, for any fixed $k \geq 1$, let $\mathfrak{M} = \{A_1 \cup \cdots \cup A_k : A_1, \ldots, A_k \text{ distinct members of } \mathcal{C}\}$. Then $\mathfrak{M} \neq \emptyset$ and $\bigcap \mathfrak{M} = \ker S$.

If $\mathcal{C}$ is the collection of all maximal convex subsets of some set $S$, it is easy to show that $\bigcap \mathcal{C} = \ker S$. This paper provides an interesting and perhaps surprising analogue of this well-known result. Throughout the paper, $\text{conv } S$, $\text{int } S$, and $\ker S$ will be used to denote the convex hull, interior, and kernel, respectively, for the set $S$.

Further, we will make use of these familiar definitions: For points $x$, $y$ in a set $S$, we say $x$ sees $y$ via $S$ if and only if the corresponding segment $[x, y]$ lies in $S$. A subset $T$ of $S$ is said to be a visually independent subset of $S$ if and only if for every $x$, $y$ in $T$, $x \neq y$, $x$ does not see $y$ via $S$.

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Theorem 1. Let $S$ be a closed subset of some linear topological space such that $\text{int} \ker S \neq \emptyset$ and $\ker S \neq S$. Let $\mathcal{C}$ denote the collection of all maximal convex subsets of $S$ and, for any fixed $k \geq 1$, let $\mathcal{M}_k = \{A_1 \cup \cdots \cup A_k : A_1, \ldots, A_k \text{ distinct members of } \mathcal{C}\}$. Then $\mathcal{M} \neq \emptyset$ and $\bigcap \mathcal{M} = \ker S$.

Proof. It is clear that $\ker S \subseteq \bigcap \mathcal{M}$, since $\ker S$ lies in every member of $\mathcal{C}$. To prove the reverse inclusion, we show that if $x \in S$ and $x \notin \ker S$, there are infinitely many distinct members of $\mathcal{C}$ which fail to contain $x$.

Since $x \notin \ker S$, we may select $p$ in $S$ with $[p, x] \notin S$. Also, select $z$ in $\text{int} \ker S \neq \emptyset$. Clearly $z$, $p$, $x$ are not collinear. Because $S$ is closed, $[p, z] \subseteq S$ and $[p, x] \notin S$, there is some point $w$ on $[z, x)$ such that $p$ sees $w$ via $S$ and $p$ sees no point of $(w, x]$ via $S$. Also, since $z \in \text{int} \ker S$, $w$ lies in the open interval $(z, x)$, and $\text{conv} \{p, z, w\} \subseteq S$. Similarly, there is a point $y$ on $(z, p)$ such that $x$ sees $y$ via $S$, $x$ sees no point of $(y, p]$ via $S$, and $\text{conv} \{x, z, y\} \subseteq S$. Let $q$ denote the point of intersection of $(p, w)$ with $(x, y)$. There are two cases to consider.

Case 1. Assume for the moment that no point of $[p, q)$ sees any point of $[x, q)$ via $S$. Consider the family of segments $[a, b]$ supporting $\text{conv} \{p, q, x\}$ at $q$, with $a$ on $[p, y)$ and $b$ on $(w, x)$. Each of these segments lies in a maximal convex subset of $S$ not containing $x$, and no two segments lie in the same maximal convex subset. Hence there are infinitely many maximal convex subsets of $S$ not containing $x$, and $x \notin \bigcap \mathcal{M}$, the desired result.

Case 2. If some point of $[p, q)$ sees some point of $[x, q)$ via $S$, select points $p_2$ and $x_2$ having this property, with $p_2$ on $[p, q)$ and $x_2$ on $(x, q)$. Clearly $p_2 \neq p$ and $x_2 \neq x$, and we may select $p_2, x_2$ so that no point of $[p, p_2)$ sees any point of $(x, x_2)$ via $S$. Repeat an earlier argument to find points $w_2$ on $[x_2, q)$, $y_2$ on $[p_2, q)$ such that $p_2$ sees $w_2$ via $S$ and $p_2$ sees no point of $(w_2, x]$ via $S$, $x_2$ sees $y_2$ via $S$ and $x_2$ sees no point of $(y_2, p]$ via $S$.

Without loss of generality, we assume that $p_2 \neq y_2$ (for otherwise the following argument may be suitably adapted using $p$, $p_2$, $x_2$ in place of $p_2$, $q_2$, $x_2$, respectively). Let $q_2$ denote the point of intersection of $[p_2, w_2]$ with $[x_2, y_2]$. It is clear that $x$ sees no point on $[p_2, q_2] \cup (x_2, q_2)$. In case no point of $[p_2, q_2)$ sees any point of $[x_2, q_2]$ via $S$, we may repeat the argument of Case 1 to obtain an infinite collection of segments supporting $\text{conv} \{p_2, q_2, x_2\}$ at $q_2$, each of which lies in a maximal convex subset of $S$ not containing $x$, and no two of which lie in the same maximal convex subset of $S$, finishing the proof.
Otherwise, some point of \([p_2, q_2]\) sees some point of \([x_2, q_2]\) via \(S\), and we repeat the previous argument to obtain points \(p_3, x_3, q_3\). Furthermore, \(x\) cannot see \(x_3\) via \(S\). Continuing inductively, if for some \(n\), no point of \([p_n, q_n]\) sees any point of \([x_n, q_n]\) via \(S\), then the argument of Case 1 yields the desired result. If no such \(n\) exists, then the infinite set of points \(\{x_{2n+1} : n \geq 1\}\) is a visually independent subset of \(S\), no point of which sees \(x\) via \(S\). To each point \(x_{2n+1}\) we may associate a distinct maximal convex subset of \(S\) not containing \(x\). Therefore, \(x \notin \bigcap \mathcal{M}\). This completes Case 2 and the proof of the Theorem.

To see that the full hypothesis of Theorem 1 is required, consider the following example.

**Example.** For \(k \geq 2\), let \(x_1, \ldots, x_k\) denote \(k\) distinct points of some line \(L\), with \(x_1 < x_2 < \cdots < x_k\), and let \(y\) be a point not on \(L\). If \(S = \text{int}(\text{conv}\{x_1, x_k, y\} \cup \{x_1, \ldots, x_k\}\), then \(S\) is not closed, \(S\) has exactly \(k\) maximal convex subsets, and the corresponding set \(\bigcap \mathcal{M}\) is all of \(S\).

Similarly, if \(S\) is any collection of \(k \geq 2\) distinct lines intersecting in a common point, then \(\text{int}(\ker S) = \emptyset\), \(S\) has exactly \(k\) maximal convex subsets, and \(\bigcap \mathcal{M} = S\).