DISK-LIKE PRODUCTS OF \( \lambda \) CONNECTED CONTINUA. I

CHARLES L. HAGOPIAN

ABSTRACT. A continuum \( X \) is \( \lambda \) connected if each two of its points can be joined by a hereditarily decomposable subcontinuum of \( X \). We prove that continua \( X \) and \( Y \) are atriodic and hereditarily unicoherent when the topological product \( X \times Y \) is disk-like. From this result and a theorem of R. H. Bing's it follows that \( \lambda \) connected continua \( X \) and \( Y \) are arc-like if and only if \( X \times Y \) is disk-like.

We call a nondegenerate metric space that is both compact and connected a continuum. Let \( X \) and \( Y \) be continua and let \( f \) be a continuous function of \( X \) onto \( Y \). If \( \epsilon \) is a positive number such that for each point \( p \) of \( Y \), the diameter of \( f^{-1}(p) \) is less than \( \epsilon \), then \( f \) is said to be an \( \epsilon \)-map of \( X \) onto \( Y \).

A continuum \( X \) is arc-like if for each \( \epsilon > 0 \) there is an \( \epsilon \)-map of \( X \) onto an arc. Arc-like continua are sometimes called snake-like or chainable. This property can be described in terms of simple chains of small open sets that cover a space [1].

A continuum \( X \) is disk-like if for each \( \epsilon > 0 \) there is an \( \epsilon \)-map of \( X \) onto a disk (2-cell).

A continuum \( T \) is called a triod if it contains a subcontinuum \( Z \) such that \( T - Z \) is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be atriodic.

A continuum is decomposable if it is the union of two proper subcontinua. A continuum is unicoherent provided that if it is the union of two subcontinua \( E \) and \( F \), then \( E \cap F \) is connected. A continuum is called hereditarily decomposable (hereditarily unicoherent) if all of its subcontinua are decomposable (unicoherent).

According to a theorem of R. H. Bing [1, Theorem 11], every atriodic, hereditarily decomposable, hereditarily unicoherent continuum is arc-like.
For any two metric spaces \((X, \psi)\) and \((Y, \phi)\), we shall always assume that the distance between two points \(p_1 = (x_1, y_1)\) and \(p_2 = (x_2, y_2)\) of the topological product \(X \times Y\) is defined by
\[
\rho(p_1, p_2) = ((\psi(x_1, x_2))^2 + (\phi(y_1, y_2))^2)^{1/2}.
\]

Throughout this paper the closure and the boundary of a given set \(Z\) are denoted by \(\text{Cl} Z\) and \(\text{Bd} Z\) respectively.

**Theorem 1.** Suppose that \(X\) and \(Y\) are continua and that the topological product \(X \times Y\) is disk-like. Then \(X\) is atriodic and hereditarily unicoherent.

**Proof.** Let \(\psi\) and \(\phi\) be distance functions for \(X\) and \(Y\), respectively, and let \(D\) be a disk in a 2-sphere \(S^2\).

Assume that \(X\) contains a triod \(T\). It follows that there exist distinct continua \(B_1, B_2, B_3,\) and \(Z\) such that \(T = \bigcup_{i=1}^3 B_i\) and \(Z = B_i \cap B_j\) for each \(i\) and \(j\) \((1 \leq i < j \leq 3)\). For \(i = 1, 2,\) and \(3\), let \(p_i\) be a point of \(B_i - \bigcup_{j \neq i} B_j\). Define \(\{y_i\mid 1 \leq i \leq 6\}\) to be a set consisting of six distinct points of \(Y\). Let \(e\) be the minimum of \(\{|\phi(y_i, y_j)|\mid 1 \leq i < j \leq 6\}\) and \(\{|\psi(p_i, B_j \cup B_k)|\mid 1 \leq i \leq 3, 1 \leq j < k \leq 3,\) and \(j \neq i \neq k\)\). Let \(f\) be an \(e\)-map of \(X \times Y\) onto \(D\).

There exist disjoint disks \(Q_1, Q_2,\) and \(Q_3\) in \(S^2\) such that for \(i = 1, 2,\) and \(3, Q_i\) contains \(f(\{p_i\} \times Y)\) and misses \(f((B_j \cup B_k) \times \{y_1\})\) when \(1 \leq j < k \leq 3\) and \(j \neq i \neq k\). By staying close to the continuum \(f((B_1 \cup B_2) \times \{y_1\})\) we define an arc-segment \(A_1\) in \(S^2 - \bigcup_{i=1}^3 Q_i\) such that each component of \(Q_1 \cup Q_2\) contains an endpoint of \(A_1\) and \(\text{Cl} A_1 \cap (\bigcup_{i=2}^6 f(T \times \{y_i\})\) = \(\emptyset\). Define \(A_2\) to be an arc-segment in \(S^2 - \bigcup_{i=1}^3 Q_i\) that stays close to \(f((B_3 \cup B_j) \times \{y_1\})\) such that \(\text{Cl} A_2\) meets \(Q_2\) and \(Q_3\) and misses \(\text{Cl} A_1 \cup \bigcup_{i=3}^6 f(T \times \{y_i\})\). Let \(A_3\) be an arc-segment in \(S^2 - \bigcup_{i=1}^3 Q_i\) near \(f((B_1 \cup B_3) \times \{y_4\})\) such that \(\text{Cl} A_3\) meets \(Q_1\) and \(Q_3\) and misses \(\text{Cl}(A_1 \cup A_2) \cup \bigcup_{i=4}^6 f(T \times \{y_i\})\).

Note that \(\bigcup_{i=1}^3 A_i \cup Q_i\) has exactly two complementary domains in \(S^2\). Hence there exists a complementary domain \(U\) of \(\bigcup_{i=1}^3 A_i \cup Q_i\) in \(S^2\) that contains two elements of \(\{f(Z \times \{y_i\})\mid 4 \leq i \leq 6\}\). Assume without loss of generality that \(f(Z \times \{y_4\})\) and \(f(Z \times \{y_5\})\) are in \(U\). Since \(Z\) is a continuum and \(f(T \times \{y_4, y_5\}) \cap (\bigcup_{i=1}^3 A_i) = \emptyset\), and since for each point \(y\) of \(Y\) and \(i = 1, 2,\) and \(3, f(B_i \times \{y\}) \cap Q_i \neq \emptyset\), it follows that there exist continua \(H\) and \(K\) in \(f(T \times \{y_4\}) \cap \text{Cl} U\) and \(f(T \times \{y_5\}) \cap \text{Cl} U\), respectively, such that for \(i = 1, 2,\) and \(3, H \cap \text{Bd} Q_i \neq \emptyset \neq K \cap \text{Bd} Q_i\). But since \(H\) and \(K\) are disjoint, this is a contradiction [6, Theorem 76, p. 220]. Hence \(X\) is atriodic.
Assume that $X$ is not hereditarily unicoherent. It follows that in $X$ there exist continua $E$ and $F$ and nonempty disjoint closed sets $A$ and $B$ such that $E \cap F = A \cup B$. Define $C_1$ and $C_2$ to be open subsets of $X$ such that $A \subset C_1$, $B \subset C_2$, and $C_1 \cap C_1 \cap C_2 = \emptyset$. Define $\delta$ to be a positive number less than $\psi(C_1, C_2)$, $\psi(E, F - (C_1 \cup C_2))$ and $\psi(F, E - (C_1 \cup C_2))$.

We first prove that $E \cup F$ is $X$. To accomplish this we suppose that there is a point $x$ of $X - (E \cup F)$. Let $R$ be a proper subcontinuum of $Y$. Let $v_1$ and $v_2$ be distinct points of $R$ and let $v_3$ be a point of $Y - R$. Define $\delta'$ to be a positive number less than $\delta$, $\psi(x, E \cup F)$, $\phi(v_1, v_2)$, and $\phi(v_3, R)$. Let $g$ be a $\delta'$-map of $X \times Y$ onto $D$.

Note that the continua $g(X \times \{v_i\})$ and $g(X \times \{v_j\})$ are disjoint for each $i$ and $j$ ($1 \leq i < j \leq 3$). Suppose that for $i = 1, 2,$ and $3$, $g((E \cup F) \times \{v_i\})$ does not separate $g(X \times \{v_j\})$ from $g(X \times \{v_k\})$ in $S^2$ when $1 \leq j < k \leq 3$ and $j \neq i \neq k$. For $i = 1$ and $2$, define $H_i$ to be an arc in $S^2 - g((E \cup F) \times \{v_i\})$ that intersects both $g(X \times \{v_j\})$ and $g(X \times \{v_k\})$ ($1 \leq j \leq 2$ and $j \neq i$).

Let $z_1$ and $z_2$ be points of $A$ and $B$ respectively. For $i = 1, 2$ and $j = 1, 2$, define $M_{ij}$ to be

$$(g((E \cap C_i) \times \{v_i\}) \cap g((F \cap C_j) \times \{v_j\}) \cup (g(z_i \times R) \cap g((E \cup F) \times \{v_i\})).$$

Note that for $j = 1$ and $2$, $M_{1j}$ and $M_{2j}$ are closed disjoint subsets of $g(C_j \times Y) - g((E \cup F) \cap C_j \times Y)$.

There exist mutually exclusive disks $K_{11}$, $K_{12}$, $K_{21}$, and $K_{22}$ in $S^2$ such that for each $i$ and $j$, the following conditions are satisfied:

1. The interior of $K_{ij}$ contains $M_{ij}$.
2. $K_{ij}$ does not intersect $H_i \cup g(((E \cup F) - C_j) \times Y) \cup g(X \times \{v_k, v_3\})$ when $1 \leq k \leq 2$ and $k \neq i$.

Let $E_1, E_2, F_1, F_2, R_1,$ and $R_2$ be disjoint continua in $S^2 - g(X \times \{v_3\})$ that miss the interior of $\bigcup_{i,j=1}^{2} K_{ij}$ such that for $n = 1$ and $2$, $E_n$ is in $g(E \times \{v_n\})$ and meets $\text{Bd} K_{n1}$ and $\text{Bd} K_{n2}$, $F_n$ is in $g(F \times \{v_n\})$ and meets $\text{Bd} K_{n1}$ and $\text{Bd} K_{n2}$, and $R_n$ is in $g(z_n \times R)$ and meets $\text{Bd} K_{n1}$ and $\text{Bd} K_{n2}$.

There exist arc-segments $I_n$, $J_n$, $I_n$, $J_n$, $T_n$, and $T_2$ in $S^2 - (g(X \times \{v_3\})$ $\cup \bigcup_{i,j=1}^{2} K_{ij})$ whose closures are disjoint approximating $E_1$, $E_2$, $F_1$, $F_2$, $R_1,$ and $R_2$, respectively, such that for $n = 1$ and $2$, the following conditions are satisfied:

1. $\text{Cl} I_n$ misses $H_n \cup g((F - (C_1 \cup C_2)) \times Y)$, meets $\text{Bd} K_{n1}$ and $\text{Bd} K_{n2}$, and contains a point $e_n$ of $E_n - g((C_1 \cup C_2) \times \{v_n\})$.
2. $\text{Cl} J_n$ misses $H_n \cup g((E - (C_1 \cup C_2)) \times Y)$, meets $\text{Bd} K_{n1}$ and $\text{Bd} K_{n2}$, and contains a point $f_n$ of $F_n - g((C_1 \cup C_2) \times \{v_n\})$.\]
3. \( \text{Cl} T_n \) misses \( g(((E \cup F) - C_n) \times Y) \) and meets \( \text{Bd} K_{1n} \) and \( \text{Bd} K_{2n} \).

Let \( V \) be the complementary domain of \( \bigcup_{i=1}^2 (L_i \cup T_i \cup \bigcup_{j=1}^2 K_{ij}) \) that contains \( g(X \times \{v_3\}) \). Note that if \( i \) and \( j \) are distinct positive integers less than 3, then the continuum \( g(X \times \{v_i, v_3\}) \cup H_{ij} \cup K_{ij} \cup K_{i2} \cup L_i \cup J_i \) misses \( K_{ij} \cup K_{i2} \cup L_j \cup J_j \). It follows that \( \text{Bd} V \) is a simple closed curve that contains \( T_1 \) and \( T_2 \) [6, Theorem 28, p. 156]. Consequently one of \( L_1, L_2, J_1, \) and \( J_2 \) does not meet \( \text{Bd} V \). Suppose, without loss of generality, that \( L_1 \cap \text{Bd} V = \emptyset \). It follows that \( \bigcup_{i=1}^2 (L_i \cup T_i \cup \bigcup_{j=1}^2 K_{ij}) \) contains a simple closed curve \( L \) that separates \( e_1 \) from \( g(X \times \{v_2\}) \) in \( S^2 \). Let \( u \) be a point of \( E - (C_1 \cup C_2) \) such that \( g(u, v_1) = e_1 \). Since \( g(\{u\} \times Y) \) is a continuum in \( S^2 - L \) that meets \( e_1 \) and \( g(X \times \{v_3\}) \), we have a contradiction. Hence for some integer \( i = 1, 2, \) or 3, the continuum \( g((E \cup F) \times \{v_i\}) \) separates \( g(X \times \{v_j\}) \) from \( g(X \times \{v_k\}) \) in \( S^2 \) when \( 1 \leq j < k \leq 3 \) and \( j \neq i \neq k \).

Assume, without loss of generality, that \( g((E \cup F) \times \{v_2\}) \) separates \( g(X \times \{v_1\}) \) from \( g(X \times \{v_3\}) \) in \( S^2 \). This assumption contradicts the fact that \( g(\{x\} \times Y) \) is a continuum in \( S^2 - g((E \cup F) \times \{v_2\}) \) that meets both \( g(X \times \{v_1\}) \) and \( g(X \times \{v_3\}) \). It follows that \( X = E \cup F \).

Next we let \( h \) be a \( \delta \)-map of \( X \times Y \) onto \( D \). Note that the set \( h(E \times Y) \cap h(F \times Y) \) lies in \( h((C_1 \cup C_2) \times Y) \) and meets both \( h(C_1 \times Y) \) and \( h(C_2 \times Y) \). Thus \( h(E \times Y) \cap h(F \times Y) \) is not connected. But since \( X = E \cup F \) and \( h(X \times Y) = D \), the union of continua \( h(E \times Y) \) and \( h(F \times Y) \) is \( D \), which contradicts the fact that \( D \) is unicoherent [6, Theorem 22, p. 175]. Hence \( X \) is hereditarily unicoherent.

**Theorem 2.** If \( X \) is a \( \lambda \) connected hereditarily unicoherent continuum, then \( X \) is hereditarily decomposable.

**Proof.** Assume that \( X \) contains an indecomposable continuum \( I \). Let \( p \) and \( q \) be points of distinct composants of \( I \) [6, Theorem 139, p. 59]. Since \( X \) is \( \lambda \) connected, there exists a subcontinuum \( H \) of \( X \) that contains \( \{p, q\} \) and does not contain \( I \). But since \( p \) and \( q \) belong to different composants of \( I \), the set \( H \cap I \) is not connected, which contradicts the assumption that \( X \) is hereditarily unicoherent. Hence \( X \) is hereditarily decomposable.

**Theorem 3.** Suppose that \( X \) and \( Y \) are continua, that \( X \) is \( \lambda \) connected, and that \( X \times Y \) is disk-like. Then \( X \) is arc-like.

**Proof.** By Theorem 1, \( X \) is atriodic and hereditarily unicoherent. Hence \( X \) is hereditarily decomposable (Theorem 2). It follows from Bing's theorem [1, Theorem 11] that \( X \) is arc-like.
Theorem 4. Suppose that $X$ and $Y$ are $\lambda$ connected continua. Then $X$ and $Y$ are arc-like if and only if $X \times Y$ is disk-like.

Proof. Theorem 3 indicates that this condition is sufficient. To see that it is also necessary we note that if $f$ is an $\varepsilon/2$-map of $X$ onto the unit interval $[0, 1]$ and $g$ is an $\varepsilon/2$-map of $Y$ onto $[0, 1]$, then the function $h$ of $X \times Y$ onto $[0, 1] \times [0, 1]$ defined by $h((x, y)) = (f(x), g(y))$ is an $\varepsilon$-map.

A continuum $X$ is said to have the fixed point property if for each continuous function $f$ of $X$ into itself there is a point $x$ of $X$ such that $f(x) = x$. It is known [3] that every $\lambda$ connected nonseparating plane continuum has the fixed point property.

Theorem 5. If $X$ and $Y$ are $\lambda$ connected continua and $X \times Y$ is disk-like, then $X$, $Y$, and $X \times Y$ have the fixed point property.

Proof. O. H. Hamilton [5] proved that every arc-like continuum has the fixed point property. In [2] E. Dyer proved that all products of arc-like continua have the fixed point property. Hence the theorem follows from Theorem 4.

For another result involving products of $\lambda$ connected continua see [4, Theorem 5].

Question 1. If $X$ and $Y$ are continua and $X \times Y$ is disk-like, then must $X$ be arc-like?

Question 2. Does every disk-like continuum have the fixed point property?

An affirmative answer to Question 2 would imply that every nonseparating plane continuum has the fixed point property.


REFERENCES


DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, SACRAMENTO, CALIFORNIA 95819