A CLASS OF \( L^p \)-BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. Pseudo-differential operators with symbol \( p(x, \xi, y) \in S^{ \mu, \delta, \epsilon}_{\rho, \delta, \epsilon} \), \( \mu \leq (\rho - 1)(n + 1) \), are proven to be generally \( L^p \)-bounded for \( 1 < p < \infty \).

Introduction. Previously \( L^2 \)-boundedness of pseudo-differential operators with symbol \( p(x, \xi, y) \in S^{ \mu, \delta, \epsilon}_{\rho, \delta, \epsilon} \) was shown by Hörmander [3] and Calderón and Vaillancourt [1], if \( \mu, \rho, \delta, \epsilon \) satisfy suitable conditions. In older theorems by Mikhlin (see [4, pp.232 ff.]) and Hörmander [2, pp. 120–123] the general \( L^p \)-boundedness for \( \mathbb{R}^n \) Fourier-multipliers with symbol \( a(\xi) \) is proven, where \( a(\xi) \) satisfies conditions closely related to those on \( p(x, \xi, y) \) in [1] and [3]. In [5], Muramatu gave a generalization of Calderón’s result and proved general \( L^p \)-boundedness for a class of pseudo-differential operators, imposing additional conditions on the Fourier transform of \( p(x, \xi, y) \). It is the purpose of this paper to replace these conditions by conditions on the symbol itself, such that \( L^p \)-boundedness of the operator is still true for \( 1 < p < \infty \).

The problem was pointed out to the author by Professor H. O. Cordes, who gave more valuable advice.

Muramatu’s notation is widely used, and parts of the proof of the main theorem are identical to the proof of his result in [5]. Other methods used in this paper are similar to Hörmander’s in [2].

Notation. \( |x| \) denotes the Euclidean norm of \( x \in \mathbb{R}^n \), and \( |x|_{\infty} = \sup |x_j| \), where \( j = 1, 2, \ldots, n \). If \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i = 0, 1, \ldots \), then \( D_1^\alpha = \partial_1^{\alpha_1} \cdots \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) and similarly \( D_x^\alpha, D_y^\alpha \) for \( x, y, \xi \in \mathbb{R}^n \). \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). \( \mathcal{S}(\mathbb{R}^n) \) is the space of \( C \)-valued rapidly decreasing functions. \( m(M) \) denotes the measure of the set \( M \subseteq \mathbb{R}^n \). We say that \( p(x, \xi, y) \) has compact support in \( \xi \) if \( \{ \xi \mid p(x, \xi, y) \neq 0 \} \) is for all \( x, y \) contained in a minimal fixed compact set, and we denote this set by \( \text{supp} \xi \ p \).

Definition. A \( C \)-valued symbol \( p(x, \xi, y) \) of \( (x, \xi, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) belongs to \( S^{ \mu, \delta, \epsilon}_{\rho, \delta, \epsilon} (\mathbb{R}^{3n}, \mathbb{C}) \) if

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\( |D_\alpha^\alpha D_\xi^\beta p(x, \xi, y)| \leq C(1 + |\xi|)^\mu + \delta|\alpha| - \rho|\beta|, \)

(2) \( |D_\gamma^\gamma D_\xi^\beta p(x, \xi, y)| \leq C(1 + |\xi|)^\mu + \epsilon|\gamma| - \rho|\beta| \)

for any multi-index \( \alpha, \beta, \gamma \), where \( 0 \leq \rho, \delta, \epsilon \leq 1 \).

We define a pseudo-differential operator \( T \) on \( \mathcal{S} \) by

(3) \( Tf(x) = (2\pi)^{-n} \int e^{ix\xi} d\xi \int p(x, \xi, y)f(y)e^{-iy\cdot} dy. \)

(1) and (2) hold if \( \rho \), \( \delta \), \( \epsilon \) for \( 0 < |\beta| < 2m, 0 < |\alpha| < 2m_1, 0 < |\gamma| < 2m_2 \), where \( m, m_1, m_2 \) are the least integers such that \( 2m > n + 2, m_1(1 - \delta'), 5n/4, m_2(1 - \epsilon') > 5n/4, \rho = \min(\rho, \max(\delta, \epsilon)), \delta' = \max(\delta, \epsilon'), \epsilon' = \max(\epsilon, \epsilon') \). Then

(a) \( \|Tf\|_{L^2} \leq C\|p\| \cdot \|f\|_{L^2} \), where \( C \) depends on \( \delta, \epsilon, \rho \) only. \( \|p\| \) denotes the least value for which (1) and (2) hold if we restrict \( |\alpha|, |\beta|, |\gamma| \) like above.

(b) If \( p \) and \( p_j \) satisfy the conditions of (a), \( p_j \) has compact support in \( \xi, \|p_j\| \) is bounded and for all \( \psi(x) \in C^\infty_0 \), \( \|p_j - p\psi\| \to 0 \) as \( j \to \infty \), the operator \( T_j \) associated to \( p_j \) as in (3) converges strongly to a limit \( T \) that depends only on \( p \), and \( \|T\| \leq C\|p\| \), with the constant \( C \) of (a).

**Proof.** See Calderón-Vaillancourt [1].

**Theorem 2** (Marcinkiewicz interpolation theorem). Let \( 1 < q < \infty \) and let \( T \) be a subadditive mapping from \( L^1(\mathbb{R}^n) + L^q(\mathbb{R}^n) \) into the space of measurable functions on \( \mathbb{R}^n \). If for all \( \lambda > 0 \)

(4) \( m\{x| |Tf(x)| > \lambda \} \leq C_1/\lambda \cdot \|f\|_{L^1} \) and

(5) \( m\{x| |Tf(x)| > \lambda \} \leq C_2/\lambda \cdot \|f\|_{L^q} \) (when \( q = \infty \) we assume \( \|Tf\|_\infty \leq C_2 \|f\|_\infty \)), we have for all \( 1 < p < q \),

\[ \|Tf\|_{L^p} \leq C_p \|f\|_{L^p}, \]

where \( C_p \) depends only on \( C_1, C_2, p \) and \( q \).

**Proof.** See, e.g., E. M. Stein [6, pp. 272–274].

(4) and (5) are called weak \( L^1 \)-boundedness and weak \( L^q \)-boundedness of \( T \) respectively. We are going to state our main result.

**Theorem 3.** Assume \( 1 < p < \infty \). Let the function \( p(x, \xi, y) \) satisfy the conditions of Theorem 1 with \( \mu \leq (\rho - 1)(n + 1) \). Then the pseudo-differential
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**Proof.** From (1) and (2) with $\mu \leq (p - 1)(n + 1)$, the reader will have no difficulty verifying that there is a constant $B$ such that

(a) $\int_{\frac{1}{2} R \leq |\xi| \leq 2 R} |R^{|a|} D^{\alpha}_\xi (p(x, \xi, y))| d\xi \leq BR^n$

for all $|a| \leq n + 1$ and all $R \geq 1$,

(b) $\int_{\frac{1}{2} R \leq |\xi| \leq 2 R} |R^{|a|} D^{\alpha}_\xi (\partial/\partial y) p(x, \xi, y)| d\xi \leq BR^{n+1}$, and

$\int_{\frac{1}{2} R \leq |\xi| \leq 2 R} |R^{|a|} D^{\alpha}_\xi (\partial/\partial x) p(x, \xi, y)| d\xi \leq BR^{n+1}$

for all $|a| \leq n + 1$, $R \geq 1$, $l = 1, \ldots, n$.

For the main part of the proof, we shall assume that $\text{supp}_\xi p$ is compact. At the end we give an argument that enables us to abandon this restriction on $p$ and thus yields the theorem's statement in the desired generality. In the whole proof, $C$ will denote constants depending on $n$ and the so far introduced constants $(B, \|p\|, C)$, but $C$ may have different values in different formulas.

So, assume that $p$ has compact support in $\xi$. In view of Theorem 1, part (a), and Theorem 2, it suffices to prove weak $L^1$-boundedness for $T$. Choose a fixed $v \in C_0^\infty(\mathbb{R}^n)$ such that $v(x) \in [0, 1]$ for all $x$, and

$$v(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| \geq 2, \end{cases}$$

and let $w = 1 - v$. Then obviously $p = pv + pw$ and $\text{supp}_\xi (pv) \subset \{\xi; |\xi| \leq 2\}$.

**Lemma 1.** The pseudo-differential operator with symbol $pv$ is $L^1$-bounded and so in particular weakly $L^1$-bounded.

**Proof.** Let $\kappa = \lceil n/2 \rceil + 1$.

$$\int |Tu(x)| dx = (2\pi)^{-n} \int \left| \int p v(x, \xi, y) e^{i(x-y)\xi} d\xi u(y) dy \right| dx$$

$$= (2\pi)^{-n} \int \left| \int \int \frac{1}{(1 + |x-y|^2)^{\kappa}} (1 - \Delta_\xi)^\kappa (pv)(x, \xi, y) e^{i(x-y)\xi} d\xi u(y) dy \right| dx$$

$$\leq (2\pi)^{-n} \int \frac{1}{(1 + |x-y|^2)^{\kappa}} \int |(1 - \Delta_\xi)^\kappa (pv)(x, \xi, y)| d\xi |u(y)| dy dx \leq C|u|_1$$
by the Fubini theorem, by $2k \geq n + 1$ and because $pv$ is bounded and has compact support in $\xi$, that is not dependent on $x, y$. □

It remains to prove that the operator with symbol $pw$ is weakly $L^1$-bounded. Note that $pw = 0$ if $|\xi| \leq 1$. As the operator with symbol $pw$ satisfies the conditions of Theorem 1 and (a) and (b) (eventually with different constants, that, however, are dependent on $p$ and the fixed $w$ only), we shall henceforth assume $pw = p$, i.e. $p$ satisfies the conditions of the theorem and $p(x, \xi, y) = 0$ if $|\xi| \leq 1$.

By Calderón-Zygmund's theorem [6, p. 17] there is for $f \in L^1$ and $\lambda > 0$ a decomposition of $\mathbb{R}^n$ so that $\mathbb{R}^n = F \cup \Omega$, $\Omega \cap \Omega = \emptyset$, $|f(x)| \leq \lambda$ a.e. on $F$, $\Omega$ is the union of cubes $\Omega = \bigcup_k Q_k$, whose interiors are disjoint, and

$$\lambda m(Q_k) \leq \|f\|_{L^1(Q_k)} \leq 2^n \cdot \lambda \cdot m(Q_k).$$

Let $Q_j^0$ denote the interior of $Q_j$. Furthermore, let

$$f_0(x) = \begin{cases} f(x), & x \in F, \\ \frac{1}{m(Q_j)} \int_{Q_j} f(y) \, dy, & x \in Q_j^0, \end{cases}$$

and let $g(x) = f(x) - f_0(x)$. By straightforward computation, we find

$$\|f_0(x)\|_{L^2}^2 \leq \lambda(1 + 2^{2^n})\|f\|_{L^1}^2,$$

and from this we get with Theorem 1

$$(6) \quad m\{x; |Tf_0(x)| > \lambda\} \leq (C/\lambda)\|f\|_{L^1}.$$  

Now, after $Tf_0$, we are going to estimate $Tg$.

**Lemma 2.** There is a function $\phi \in C_0^\infty$ with support in $\{1/2 < |\xi| < 2\}$ such that $\sum_{j=0}^{\infty} \phi(2^{-j} \xi) = 1$ if $|\xi| \geq 1$.

**Proof.** Let $\phi \geq 0$ be in $C_0^\infty$ with support in $\{1/2 < |\xi| < 2\}$ and $\phi(\xi) > 0$ if $1/\sqrt{2} < |\xi| < \sqrt{2}$. Set

$$\phi(\xi) = \phi(\xi) \sum_{j=-\infty}^{\infty} \phi(2^{-j} \xi).$$

Then the denominator is never 0 for $\xi \neq 0$, and so $\phi \in C_0^\infty$. If $|\xi| > 1$, we have $|2^k \xi| > 2$ ($k > 0$), and this proves the lemma. □

Let $p_j(x, \xi, y) = p(x, \xi, y) \cdot p(2^{-j} \xi)$ and notice that $\text{supp}_{\xi} p_j \subset \{2^{j-1} <$
\[ |\xi| < 2^{j+1} \] for \( j = 0, 1, 2, \ldots \). Let \( Q \) be an arbitrary, nonempty cube with sides parallel to the axes. Let \( 2r \) be the length of the side of \( Q \), \( s = \sqrt{n}r, x_0 \) the center of \( Q \), \( B = \{ x; |x - x_0| \leq 2s \} \). Furthermore, let \( u \) be a function with support in \( Q \) such that \( \int u = 0 \). We shall show

\[
\int_{|x-x_0| \geq 2s} \left| \int p(x, \xi, y)u(y)e^{i(x-y)\xi} \, dy \, d\xi \right| \, dx \leq C \int |u| \, dx
\]

with a constant \( C \) dependent neither on \( s \) nor on \( \text{supp} \xi \). p.

To do so, use the Leibniz' formula to get

\[ D_\xi^a p_j(x, \xi, y) = \sum_{\beta + \gamma = a} 2^{-j|\beta|}(D_\xi^\beta)p(x, \xi, y)(D_\phi^\gamma)(2^{-j} \xi), \]

and since the derivatives of \( \phi \) are bounded, we get from (a) with \( R = 2^j \):

\[
\int_{2^{-j-1}|\xi| \leq 2^{j+1}} |2^j a| D_\xi^a p_j(x, \xi, y) \, d\xi \leq CB2^{nj}.
\]

The next estimate has to be done differently for \( n \) odd and \( n \) even.

(A) \( n \) odd. Pointwise in \( x \), we use partial integration in \( \xi \) in the inner integral to compute

\[
\int_{|x-y| \geq 2s} (1 + 2^{j(n+1)}|x-y|^{n+1}) \left| \int p_j(x, \xi, y)e^{i(x-y)\xi} \, d\xi \right|^2 \, dx
\]

\[
= \int_{|x-y| \geq 2s} \frac{1}{|x-y|^{n+1}} \left| \int (D_\xi^a p_j(x, \xi, y)e^{i(x-y)\xi} + 2^{j((n+1)/2)}D_\xi^\alpha p_j(x, \xi, y)e^{i(x-y)\xi}) \, d\xi \right|^2 \, dx,
\]

where \( |\alpha| = (n + 1)/2 \), \( |\alpha'| = n + 1 \), and \( \alpha = (0, 0, \ldots, 0, |\alpha|, 0, \ldots, 0) \), \( \alpha' = (0, \ldots, |\alpha'|, \ldots, 0) \), the place of \( |\alpha| \), \( |\alpha'| \) depending on \( x \). But (9) is less than

\[
(C/s)(2^{-j((n+1)/2)}CB2^{nj})^2 = (C/s)(B2^{j(n-1)/2})^2
\]

by the equivalence of \( \| \| \) and \( \| \|_\infty \) and by (8). The Cauchy-Schwarz inequality now yields

\[
\int_{|x-y| \geq 2s} \left| \int p_j(x, \xi, y)e^{i(x-y)\xi} \, d\xi \right| \, dx
\]

\[
\leq CB \cdot 2^{j(n-1)/2} \sqrt{s} \left( \int \frac{dx}{1 + 2^{j(n+1)}|x-y|^{n+1}} \right)^{1/2} = CB(2^j s)^{-1/2}.
\]
The above estimate for \( \int_{|x-y| \geq s} |\int p_j(x, \xi, y) e^{i(x-y)\xi} \, d\xi| \, dx \) is good if \( 2^{i}s \) is big. For small \( 2^{i}s \) we use another estimate. For this, observe that because \( \int u = 0 \), we can, to obtain (7), replace \( \int_{|x-y| \geq s} |\int p_j(x, \xi, y) e^{i(x-y)\xi} \, d\xi| \, dx \) by

\[
K_j(x_0, y, s) = \int_{|x-x_0| \geq 2s} \left| \int p_j(x, \xi, y) e^{i(x-y)\xi} \, d\xi \right| dx
\]

Because \( \text{supp } u \subset Q \), \( |x - x_0| \geq 2s \) implies \( |x - y| \geq s \), and so we find like above

(11)

\[
K_j(x_0, y, s) \leq C B (2^{i}s)^{-1/2}.
\]

Now let \( y(t) = y + t(x_0 - y), t \in [0, 1] \). Then

\[
K_j(x_0, y, s) = \int_{|x-x_0| \geq 2s} \left| \int \nabla_y (p_j(x, \xi, y(t)) e^{i(x-y(t))\xi}) (x_0 - y) \right| dt d\xi dx
\]

\[
\leq \int_{t=0}^1 \int_{|x-x_0| \geq 2s} \left| \sum_{m=1}^n \int D_{y_m} (p_j(x, \xi, y(t)) e^{i(x-y(t))\xi}) (x_0 - y_m) \right| dx dt d\xi dx dt,
\]

because \( |x_{0m} - y_m| \leq s \). The integral over \( x \) is now estimated exactly like above. Observing that \( |\xi| \leq 2^{i+1} \) and using just (b) and the ordinary differentiation rules, we find

\[
\sum_{m=1}^n \int_{t=0}^1 \int_{|x-x_0| \geq 2s} \left| \int D_{y_m} (p_j(x, \xi, y(t)) e^{i(x-y(t))\xi}) \right| dx dt d\xi dx dt,
\]

i.e.

(12)

\[
K_j(x_0, y, s) \leq (CB/s)^{2i/2},
\]

which is a good estimate whenever \( 2^{i}s \) is small.

(B) \( n \) even. We estimate again \( K_j(x_0, y, s) \), obtaining the same result.

The estimate works in almost the same way, except: (1) The function 1 +
$2^{(n+1)}|x-y|^{n+1}_{\infty}$ has to be replaced by $f = |x-y|_{\infty}(2^i + 2^{i(n+1)}|x-y|^{n}_{\infty})$.

(2) We obtain $1/|x-y|^{n+2}_{\infty}$ in the integral over $x$ by differentiating $(n/2 + 1)$ times in the inner integral. One power of $|x-y|_{\infty}$ cancels the factor $|x-y|_{\infty}$ in $f$. We end up with the same result. So we have for all dimensions $n$

$$K_j(x_0, y, s) \leq CB \cdot \min\{2^{-s}, (2^i s)^{1/2}\}.$$ 

and the estimate is valid for all $x_0$ and all functions $u$ satisfying the stated conditions. Let

$$q_j(x, y) = \int_{\xi} \hat{p}_j(x, \xi, y)e^{i(x-y)\xi} d\xi,$$

and let $Q_N = \sum_{j=0}^{N} q_j$. Using the triangle inequality and (13), we find

$$\int_{|x-x_0| \geq 2s} |Q_N(x, y) - Q_N(x, x_0)| dx$$

$$\leq CB \sum_{j=0}^{N} \min\{2^{-s}, (2^i s)^{1/2}\} \leq C,$$

since the sum is a bounded function in $s$. The constant does not depend on $N$. If we set $P_N = \sum_{j=0}^{N} p_j$, we get $p = P_N$ for all $N \geq N_0$, where $N_0$ depends on the support of $p$. Notice that $P_N$ yields the operator kernel $Q_N$.

**Lemma 3.** Let $h(x, y)$ be a kernel function for an integral operator $H$ defined by $Hu(x) = \int h(x, y)u(y) dy$, define $h_y(x) = h(x, y)$ and assume $h_y \in L^1$ for all $y$. Then

$$\|Hu\|_{L^1} \leq \left(\esssup_y \|h_y\|_{L^1}\right) \cdot \|u\|_{L^1}.$$ 

The proof is a simple application of the Fubini theorem. □

As an immediate consequence of the lemma and estimate (14), we have

$$\int_{|x-x_0| \geq 2s} \int_{|y-x_0| \leq s} \int (P_N(x, \xi, y)e^{i(x-y)\xi} - P_N(x, \xi, x_0)e^{i(x-x_0)\xi})u(y) d\xi dy \leq C \int |u| dy.$$ 

Now remember the definition of $g$ and $Q_k$. Let $x^k$ be the center of $Q_k$, $2b_k$ the length of the side of $Q_k$, $r_k = \sqrt{n} b_k$, $B_k = \{x; |x-x^k| \leq 2r_k\}$, $D' = \bigcup_k B_k$, $D = \mathbb{R}^n \setminus D'$,

$$g_k(x) = \begin{cases} g(x), & x \in Q_k' \setminus D' \\ 0, & \text{otherwise.} \end{cases}$$
Notice that \( \int g_k = 0 \) and that \( \text{supp } g_k \subseteq Q_k^0 \). (7) implies, if \( N \geq N_0 \),
\[
\int_D |Tg_k(x)| \, dx \leq C \|g_k\|_{L^1},
\]
where the constant \( C \) is not dependent on \( k \). So
\[
\int_D |Tg(x)| \, dx \leq C \int |g(x)| \, dx,
\]
and as \( g = f - f_0 \), we find
\[
\int |g(x)| \, dx \leq C(\|f\|_{L^1} + 2^n \lambda m(\Omega)) \leq C \cdot (1 + 2^n) \|f\|_{L^1},
\]
i.e.
\[
\int_D |Tg(x)| \, dx \leq C \cdot \|f\|_{L^1}.
\]
This implies \( m(D \cap \{ |Tg(x)| > \lambda \}) \leq (C/\lambda) \|f\|_{L^1} \). Since \( m(D') \leq C m(\Omega) \), we have
\[
(15) \quad m\{x; |Tg(x)| > \lambda\} \leq (C/\lambda) \|f\|_{L^1}.
\]
So, because
\[
m\{x; |Tf(x)| > 2\lambda\} \leq m\{x; |Tf_0(x)| > \lambda\} + m\{x; Tg(x) > \lambda\},
\]
we get from (6) and (15) the weak \( L^1 \)-boundedness for \( T \). The strong \( L^p \)-boundedness for \( 1 < p \leq 2 \) follows immediately from this and Theorem 1 (a), as an application of Theorem 2.

For \( 2 < p < \infty \), let \( 1/q = 1 - 1/p \). Then \( 1 < q < 2 \). We have for \( f, g \in \mathcal{S}(\mathbb{R}^n) \)
\[
|\int (Tf)(x)g(x) \, dx| = |\int (2\pi)^{-n} \int p(x, \xi, y)f(y)e^{i(x-y)\xi} \, dy \, d\xi g(x) \, dx|
\]
\[
= |\int (2\pi)^{-n} \int p(x, \xi, y)g(x)e^{i(x-y)\xi} \, dx \, d\xi f(y) \, dy|
\]
\[
\leq \|f\|_{L^p} \|T^*g\|_{L^q},
\]
where \( T^* \) is the adjoint operator of \( T \). If \( T \) is weakly \( L^1 \)-bounded with constant \( C \), so is \( T^* \). As \( T^* \) is strongly \( L^2 \)-bounded, we conclude as before
\[
|\int (Tf)(x)g(x) \, dx| \leq \|f\|_{L^p} \|T^*g\|_{L^q} \leq C_q \|f\|_{L^p} \|g\|_{L^q},
\]
and this proves general \( L^p \)-boundedness of \( T \).

To abandon the restriction on \( \text{supp } \xi \subseteq p \), we can use methods identical to
those of the proof of Theorem 1(b), so the details are left to the reader. The statement of the theorem turns out to be valid for all $p$ satisfying our conditions.

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