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A LARGE CLASS OF SMALL VARIETIES OF LATTICE-ORDERED GROUPS

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ABSTRACT. We establish the existence of a countable collection of varieties, each of which covers the abelian variety in the lattice of varieties of lattice-ordered groups.

For each positive integer n, let $\mathfrak{L}n$ denote the variety of lattice-ordered groups (hereafter, l-groups) satisfying the law $x^n y^n = y^n x^n$. Obviously, $\mathfrak{L}1 = \mathfrak{L}A$, the variety of abelian l-groups, and $\mathfrak{L}n$ is contained in $\mathfrak{L}m$ if n is a divisor of m. Martinez [4] notes that the containment $\mathfrak{L}n \subseteq \mathfrak{L}m$ is proper if n is a proper divisor of m by considering a type of example which fails to be commutative or representable in intuitively the least way: the examples are each generated by two noncommuting positive elements a and x such that distinct conjugates of a by powers of x are disjoint. Thus, one is led to suspect that because the examples are barely nonabelian, the varieties they generate might be minimally nonabelian. It will facilitate the discussion to describe these examples as subgroups of a wreath product of ordered permutation groups. (For background, definitions, etc. concerning ordered permutation groups, the reader is referred to [1], [2].)

Let $\mathbf{Z} \wr \mathbf{Z}$ be the ordered wreath product of two copies of the integers, and define

$$G_n = \{((\ldots, w(z), \ldots), \bar{u}) \in \mathbf{Z} \wr \mathbf{Z} \mid i \equiv j \pmod{n} \Rightarrow w(i) = w(j)\}.$$

Then $G_n \in \mathfrak{L}n$. Let a, x denote the elements of G_n given by:

$$\bar{a} = 0, \qquad a(z) = \begin{cases} 1 & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{if } z \not\equiv 0 \pmod{n}, \end{cases}$$

$$\bar{x} = 1, \qquad x(z) = 0 \quad \forall z \in \mathbf{Z}.$$

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Then $\overline{a^m x^m} = m = \overline{x^m a^m}$, and

$$(a^m x^m)(z) = \begin{cases} m & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{if } z \not\equiv 0 \pmod{n}, \end{cases}$$

$$(x^m a^m)(z) = (x^m)(z) + (a^m)(z+m) = (a^m)(z+m)$$
$$= \begin{cases} m & \text{if } z + m \equiv 0 \pmod{n}, \\ 0 & \text{if } z + m \not\equiv 0 \pmod{n}. \end{cases}$$

Hence, $x^m a^m = a^m x^m$ iff $m \equiv 0 \pmod{n}$, iff $n | m$. Consequently, if n is a proper divisor of m, then m does not divide n, so $G_m \notin \mathfrak{L}n$. It also follows, as Martinez suggests, that if $\mathfrak{L}n$ is contained in $\mathfrak{L}m$, then n is a factor of m since

$$\mathfrak{L}n \subseteq \mathfrak{L}m \Rightarrow G_n \in \mathfrak{L}m \Rightarrow a^m x^m = x^m a^m \Rightarrow n \mid m.$$

It seems apparent that G_n is a minimal nonabelian l-group satisfying $x^n y^n = y^n x^n$. More precisely, we shall show that if n is prime, the variety of l-groups generated by G_n is minimal with respect to containing nonabelian members, by showing that every nonabelian, subdirectly irreducible l-group in $\mathfrak{L}n$ contains an l-subgroup l-isomorphic to G_n.

Lemma 1. *If C is a convex l-subgroup of $G \in \mathfrak{L}n$, then $x^{-n} C x^n = C$ for all $x \in G$.*

Proof. Suppose $1 < c \in C$. Then

$$1 < c < c^2 < \cdots < c^n = x^{-n} c^n x^n \in x^{-n} C x^n,$$

so $c \in x^{-n} C x^n$, and hence $C \subseteq x^{-n} C x^n$. Similarly, $C \subseteq x^n C x^{-n}$, so $C = x^{-n} C x^n$.

Lemma 2. *If C is a convex l-subgroup of $G \in \mathfrak{L}n$ and $x \in G$, then the number of distinct conjugates of C of the form $x^{-i} C x^i$ is a divisor of n.*

Proof. Let i be the smallest positive integer such that $x^{-i} C x^i = C$. If i is not a divisor of n, then there are integers r and s such that $rn + si = k$, where $1 \le k < i$, and k is the greatest common divisor of n and i. Then $x^{-k} C x^k = x^{-rn-si} C x^{rn+si} = C$, contradicting the minimality of i.

Lemma 3. *Let $\mathfrak{L}R$ denote the variety of regular l-groups. For any positive integer n, $\mathfrak{L}R \cap \mathfrak{L}n = \mathfrak{L}A$ [4, §6.4].*

Holland has shown [2] that every l-group is l-isomorphic to an l-subgroup of the l-group of all order-preserving permutations of some chain, and that an l-group G has a transitive representation as an l-group of order-preserving permutations of a chain if and only if G contains a *representing subgroup*, i.e., a convex prime l-subgroup which contains no nontrivial l-ideal of G. Among those l-groups which have transitive representations are all subdirectly irreducible l-groups. These facts, together with the preceding lemma, yield the following result.

Theorem 1. *If m and n are relatively prime, then $\mathfrak{L}m \cap \mathfrak{L}n = \mathfrak{L}A$.*

Proof. Let G be a subdirectly irreducible member of $\mathfrak{L}m \cap \mathfrak{L}n$, and let S be a chain on which G acts transitively. Suppose $g \in G$ fixes some $x \in S$. Choose integers r and s such that $rm + sn = 1$. If h is any member of G such that $xh \neq x$, then

$$xh = xg^{mn}h^{rm+sn} = xh^{rm}g^{mn}h^{sn} = xh^{rm+sn}g^{mn} = xhg^{mn}.$$

Therefore, g fixes xh, and since h is arbitrary and G is transitive on S, $g = 1$. Hence, $G_x = \{1\}$, so G is totally ordered, and thus regular. But $\mathfrak{L}R \cap \mathfrak{L}m = \mathfrak{L}A$, so G is abelian. $\mathfrak{L}m \cap \mathfrak{L}n$ is generated by its subdirectly irreducible members, and these are all abelian, so $\mathfrak{L}m \cap \mathfrak{L}n = \mathfrak{L}A$.

Lemma 4. *If C is a representing subgroup of $G \in \mathfrak{L}n \backslash \mathfrak{L}A$, then there exists $1 < x \in G$ such that $x^{-1}Cx \neq C$.*

Proof. Suppose $x^{-1}Cx = C$ for all $1 < x \in G$. Then C is an l-ideal of G, so since it is also a representing subgroup, $C = \{1\}$, and G is totally ordered, therefore regular. But $\mathfrak{L}R \cap \mathfrak{L}n = \mathfrak{L}A$, so G must be abelian. This is a contradiction; therefore, there is a positive x for which $x^{-1}Cx \neq C$.

Lemma 5. *If C is a representing subgroup of $G \in \mathfrak{L}n$ which has n distinct conjugates of the form $x^{-i}Cx^{i}$ for some $1 < x \in G$, then G contains an l-subgroup l-isomorphic to G_n.*

Proof. Suppose we have G, C, x with these properties. We shall find $a_0 \in G$ such that a_0 and x correspond under the isomorphism to the two generating elements of G_n denoted a and x in the discussion above.

For $0 \leq i \leq n-1$, define

$$C_i = x^{-i}Cx^{i}, \qquad D_i = \bigcap_{j \neq 1} C_j.$$

Let S be the chain of right cosets of C_0, and let s_0 denote the coset C_0 in this chain. For $0 < i < n-1$, define $s_i = s_0 x^{i}$, so that C_i is the stabilizer of s_i and D_i consists of all permutations of S in G which fix the set $\{s_j | j \neq i\}$. Since by hypothesis $C_0 \neq C_i$, there exists a positive permutation $h_i \in C_0 \setminus C_i$, $i = 1, \ldots, n-1$, whence

$$1 < g_0 = h_1 h_2 \cdots h_{n-1} \in C_0 \Big\backslash \bigcup_{i=1}^{n-1} C_i.$$

Define

$$d_0 = x \wedge \left(\bigwedge_{i=1}^{n-1} x^{-i} g_0^n x^i \right).$$

Since g_0 fixes only s_0 among $s_0, \ldots, s_{n-1}$, $x^{-i} g_0^n x^i$ fixes only s_i among $s_0, \ldots, s_{n-1}$. Thus d_0 moves only s_0 among $s_0, \ldots, s_{n-1}$: $1 < d_0 \in D_0 \setminus \bigcup_{i=1}^{n-1} D_i$. Also, $d_0 < x$ since x moves s_0. Define $d_i = x^{-i} d_0 x^i$, $i = 1, \ldots, n-1$. Then $1 < d_i \in D_i \setminus \bigcup_{j \neq i} D_j$. By convexity, for any i, $0 \leq i \leq n-1$,

$$e_i = \bigvee_{j \neq i} (d_i \wedge d_j) \in \bigcap_{k=0}^{n-1} D_k = \bigcap_{k=0}^{n-1} C_k.$$

Put $a_i = x^{-i} a_0 x^i = d_i e_i^{-1}$, $i = 1, \ldots, n-1$. Then $x^{-n} a_i x^n = a_i$, $1 < a_i < x$, $a_i \in D_i \setminus \bigcup_{j \neq i} D_j$, $a_i \wedge a_j = 1$ if $i \neq j$, as may be routinely checked. As a consequence of the latter, $a_i a_j = a_j a_i$ $\forall i, j$.

For any $u = ((\ldots, u(z), \ldots), \bar{u}) \in G_n$, define

$$u\theta = a_0^{u(0)} a_1^{u(1)} \cdots a_{n-1}^{u(n-1)} x^{\bar{u}}.$$

It is easily verified that θ is a group homomorphism.

That θ is a lattice homomorphism follows from the fact that in G,

$$1 \vee \left(a_0^{u(0)} \cdots a_{n-1}^{u(n-1)} x^{\bar{u}} \right) = \begin{cases} 1 & \text{if } \bar{u} < 0, \\ a_0^{u(0) \vee 0} \cdots a_{n-1}^{u(n-1) \vee 0} & \text{if } \bar{u} = 0, \\ a_0^{u(0)} \cdots a_{n-1}^{u(n-1)} x^{\bar{u}} & \text{if } \bar{u} > 0, \end{cases} \tag{1}$$

which we prove as follows. Since $a_i \wedge a_j = 1$ for $i \neq j$, $\prod_{i \in T} a_i^{\alpha(i)} \wedge \prod_{i \notin T} a_i^{\alpha(i)} = 1$ where $\alpha(i) \geq 0$ for each i and $T \subseteq \{0, 1, \ldots, n-1\}$. Given $a_0^{u(0)} \cdots a_{n-1}^{u(n-1)}$, let $T = \{i | u(i) \geq 0\}$. Then

$$\begin{aligned} \left(a_0^{u(0)} \cdots a_{n-1}^{u(n-1)} \right) \vee 1 &= \prod_{i \in T} a_i^{u(i)} \left(\prod_{i \notin T} a_i^{u(i)} \vee \prod_{i \in T} a_i^{-u(i)} \right) \\ &= \prod_{i \in T} a_i^{u(i)} \left(\prod_{i \notin T} a_i^{-u(i)} \wedge \prod_{i \in T} a_i^{u(i)} \right)^{-1} \\ &= \prod_{i \in T} a_i^{u(i)} = a_0^{u(0) \vee 0} \cdots a_{n-1}^{u(n-1) \vee 0}. \end{aligned}$$

Next we show $a_0^m < x$ for all $m \in \mathbf{Z}$. Since $x > a_0 > 1$, this is so for $m \leq 1$.

$$a_0^m < x \Leftrightarrow (d_0e_0^{-1})^m < x \Leftrightarrow \left(\bigwedge_{j=1}^{n-1} (1 \vee d_0 d_j^{-1})\right)^m < x \Leftrightarrow \left(1 \vee \left(\bigwedge_{j=1}^{n-1} d_0 d_j^{-1}\right)\right)^m < x,$$

so it suffices to show $f^m \leq x$ for $m \geq 2$, where $f = \bigwedge_{j=1}^{n-1} d_0 d_j^{-1}$. The induction hypothesis is that $f^k < x$ for $0 \leq k < m$. f^m is an infimum of terms, one of which is $d_0 d_{n-1}^{-1} f^{m-2} d_0 d_1^{-1}$. But

$$d_0 d_{n-1}^{-1} f^{m-2} d_0 d_1^{-1} = d_0 x^{1-n} d_0^{-1} x^{n-1} f^{m-2} d_0 x^{-1} d_0^{-1} x < x$$

iff

$$f^{m-2} < x^{1-n} d_0 x^{n-1} d_0^{-1} x x^{-1} d_0^{-1} = x,$$

since $x^n d_0^{-1} = d_0^{-1} x^n$. Therefore, $f^m < x$, so $a_0^m < x$ for all $m \in \mathbf{Z}$. By conjugation $a_i^m < x$ for any $m \in \mathbf{Z}$, $i = 0, 1, \ldots, n-1$, so $x^{-1} < a_0^{u(0)} \cdots a_{n-1}^{u(n-1)} < x$ for any $u(0), \ldots, u(n-1) \in \mathbf{Z}$, and hence

$$a_0^{u(0)} \cdots a_{n-1}^{u(n-1)} x^{-\bar{u}} < 1 < a_0^{u(0)} \cdots a_{n-1}^{u(n-1)} x^{\bar{u}}$$

if $\bar{u} > 0$, for any $u(0), \ldots, u(n-1) \in \mathbf{Z}$. Thus, (1) holds.

θ is 1-1 since the element a of G_n given by

$$\bar{a} = 0, \quad a(z) = \begin{cases} 1 & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{if } z \not\equiv 0 \pmod{n} \end{cases}$$

belongs to the smallest nontrivial l-ideal of G_n, but $a\theta = a_0 \neq 1$. This completes the proof of the lemma.

Theorem 2. *If n is a prime, the variety of l-groups $[G_n]$ generated by G_n covers $\mathfrak{L}A$ in the lattice of varieties of l-groups.*

Proof. If G is a subdirectly irreducible l-group in $\mathfrak{L}n$ but not in $\mathfrak{L}A$, then by Lemma 3, G contains a representing subgroup C which is not an l-ideal of G. Thus, there exists $1 < x \in G$ such that $x^{-1}Cx \neq C$. Since n is prime, it follows from Lemma 2 that C has n distinct conjugates of the form $x^{-i}Cx^i$, so by Lemma 5, G contains an l-subgroup l-isomorphic to G_n. Therefore, any variety of l-groups contained in $\mathfrak{L}n$ and which has nonabelian members contains the variety $[G_n]$. Hence, $[G_n]$ covers $\mathfrak{L}A$ in the lattice of varieties of l-groups if n is prime.

BIBLIOGRAPHY

1. P. M. Cohn, *Universal algebra*, Harper and Row, New York, 1965. MR 31 #224.

2. W. C. Holland, *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J. 10 (1963), 399–408.

3. W. C. Holland and S. H. McCleary, *Wreath products of ordered permutation groups*, Pacific J. Math. 31 (1969), 703–716. MR 41 #3350.

4. Jorge Martinez, *Free products in varieties of lattice-ordered groups*, Czechoslovak Math. J. 22 (1972), 535–553.

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