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ARTINIAN IDEALS IN NOETHERIAN RINGS

T. H. LENAGAN

ABSTRACT. We show that in a right and left Noetherian ring two-sided ideals which are Artinian as left submodules are also Artinian as right submodules.

It is well known that in a semiprime ring the right and left socles coincide. It is also standard knowledge that a right and left Noetherian ring which is left Artinian is also right Artinian. The result stated in the Abstract, although in the same vein as these results, seems to have been overlooked. This result follows easily from the Proposition of this note. We assume that rings have identity elements. When we say that a module has *finite length*, we mean that it has a finite composition series whose factors are simple modules. We refer the reader to [1] for basic facts concerning Noetherian rings.

Proposition. *Let R be a right Noetherian ring and I a two-sided ideal of R which has finite length as a left R-module. Then I has finite length as a right R-module.*

Proof. By induction on the composition length of ${}_RI$, we may assume that I is a minimal nonzero two-sided ideal. The set $r(I) = \{r \in R \mid Ir = 0\}$ is then a prime ideal. Therefore, by Goldie's theorem, the ring $S = R/r(I)$ has a simple Artinian right quotient ring Q. Obviously, I is a minimal R-S-bimodule. Let Γ denote the set of non-zero-divisors in S. Note that, since ${}_RI$ has finite length, an R-endomorphism of ${}_RI$ is a monomorphism if and only if it is an epimorphism.

Suppose that for some $c \in \Gamma$ there is a nonzero element $i \in I$ with $ic = 0$. Since, by Goldie's theorem, a non-zero-divisor in a right Noetherian prime ring generates an essential right ideal, we see that the singular submodule, $Z(I) = \{i \in I \mid \mathrm{ann}_S(i)$ is an essential right ideal of $S\}$, is nonzero.

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However, $Z(I)$ is an R-S-bimodule and so $Z(I) = I$. Since $_R I$ has finite length, there exist elements $i_1, \ldots, i_n$ of I with $I = Ri_1 + \cdots + Ri_n$. Put $K_j = \operatorname{ann}_S(i_j)$ and $K = \bigcap_{j=1}^n K_j$. Then $IK = 0$ and $K \neq 0$, a contradiction.

Therefore, for each $c \in \Gamma$, the R-endomorphism $I \longrightarrow Ic$ is a monomorphism and hence an epimorphism. But this means that we can give I a Q-module structure by defining $ic^{-1} = i'$ where $i'c = i$. Since I_S is Noetherian, I_Q is certainly Noetherian and is therefore Artinian. Let N_Q be a minimal submodule of I_Q. Then Q, being simple Artinian, is isomorphic to a finite direct sum of copies of N. However, N is a Noetherian S-module and so Q is a Noetherian S-module. If $c \in \Gamma$, the ascending chain $S \subseteq c^{-1}S \subseteq c^{-2}S \subseteq \cdots \subseteq Q$ must become stationary; suppose that $c^{-n}S = c^{-n-1}S$. Then $cS = S$ and $c^{-1} \in S$. Hence $S = Q$ and so S is Artinian. Thus I_S is Artinian and I_R is Artinian. Since I_R is also Noetherian, I_R has finite length. $\square$

REFERENCE

1. A. W. Goldie, *The structure of Noetherian rings*, Lectures on Rings and Modules, Lecture Notes in Math., vol. 246, Springer-Verlag, New York, 1972.

20 CHAMBERS STREET, EDINBURGH, SCOTLAND