SOME BAIRE SPACES FOR WHICH BLUMBERG'S
THEOREM DOES NOT HOLD

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ABSTRACT. First, in the second section, we describe a class of Baire
spaces for which Blumberg's theorem does not hold. Then, in the third sec-
tion, we discuss Blumberg's theorem for P-spaces.

1. In [2], J. C. Bradford and C. Goffman proved that a metrizable space
X is a Baire space if and only if the following statement, called Blumberg’s
theorem, holds.

1.1 If \( f \) is a real valued function defined on \( X \), then there is a dense
subset \( D \) of \( X \) such that \( f|D \) is continuous.

It follows from their proof that every topological space for which 1.1
holds is a Baire space. In [15], the author gave several examples of com-
pletely regular, Hausdorff, Baire spaces for which, if \( 2^{\aleph_0} = \aleph_1 \), 1.1 does not
hold (see also [13], [14]). In §2 we establish, using a lemma from [14], a re-
sult which shows that there are a number of Baire spaces for which 1.1 does
not hold.

2. For any topological space \( X \), we denote the weight of \( X \), the pseudo-
weight of \( X \), the density character of \( X \), and the ring of all bounded real val-
ued, continuous functions defined on \( X \) by \( wX \), \( mwX \), \( dX \), and \( C^*(X) \), respec-
tively (see [4, p. 619]). For any subset \( A \) of \( X \), we denote the closure of \( A \)
by \( cl A \). We denote the set of all real numbers by \( R \).

2.1 Theorem. Suppose \( X \) is a Baire space of cardinality \( 2^{\aleph_0} \) such that
(a) \( X \) satisfies the countable chain condition,
(b) \( wX = dX = 2^{\aleph_0} \), and
(c) every set of the first category in \( X \) is nowhere dense in \( X \).

Then 1.1 does not hold for \( X \).

Proof. Let \( B \) denote a base for the topology \( T \) on \( X \) of cardinality \( 2^{\aleph_0} \)
such that \( \emptyset, X \in B \). We may assume that \( B \) is closed under countable union.

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Let \( \mathbb{M} \) denote the set of all real valued functions defined on \( X \) that are measurable (\( \mathcal{S} \)), where \( \mathcal{S} \) is the \( \sigma \)-algebra generated by \( \mathcal{B} \). We shall now prove the following statement.

2.2 Suppose \( f: X \to R \) and there is a dense subset \( D \) of \( X \) such that \( f|D \) is continuous. Then there is \( g \) in \( \mathbb{M} \) such that \( \{ x \in X : f(x) = g(x) \} \) is dense in \( X \).

Because (a) holds, there is a function \( \gamma: \mathcal{T} \to \mathcal{B} \) such that for each \( U \) in \( \mathcal{T} \), \( \gamma(U) \) is a dense subset of \( U \). By Lemma 1.1 of [14] (see also [6, p. 202]), there is a \( G_δ \) set \( K \) containing \( D \) and a continuous, real valued function \( h \) defined on \( K \) such that \( h|D = f|D \). Let \( \{ V_n : n \in N \} \) (\( N \) denotes the set of natural numbers) denote a base for the usual topology on \( R \). For each \( n \in N \), choose \( U_n \) in \( \mathcal{T} \) such that \( h^{-1}[V_n] = U_n \cap K \). Suppose \( K = \bigcap \{ G_n : n \in N \} \), where each \( G_n \) is open. Let \( C \) denote the union of \( \bigcup \{ X \sim \gamma(G_n) : n \in N \} \) and \( \bigcup \{ U_n \sim \gamma(U_n) : n \in N \} \), and let \( W = \gamma(X \sim cl C) \). Then, because (c) holds, \( W \) is dense in \( X \). It is easily checked that \( W \subseteq K \) and, for each \( n \) in \( N \), \( (h|W)^{-1}[V_n] = \gamma(U_n) \cap W \). So, if we define \( g \) by letting

\[
g(x) = \begin{cases} h(x) & \text{if } x \in W, \\ 0 & \text{if } x \in X \sim W, \end{cases}
\]

then \( g \in \mathbb{M} \). And the set \( \{ x \in X : f(x) = g(x) \} \) is dense in \( X \) because it contains \( D \cap W \).

Because \( |\mathcal{B}| = 2^{\aleph_0} \), the cardinality of \( \mathcal{S} \) is \( 2^{\aleph_0} \) [9, p. 26, exercise 9]. Now if \( h \in \mathbb{M} \), then there is a sequence \( \{ h_n : n \in N \} \) of functions in \( \mathbb{M} \) such that each \( h_n \) has finite range and \( \lim_{n \to \infty} h_n(x) = h(x) \) for all \( x \) in \( X \). Therefore \( |\mathbb{M}| = 2^{\aleph_0} \). The proof of Theorem 2.1 will be completed if we prove the following statement.

2.3 There is a function \( f: X \to R \) such that if \( g \in \mathbb{M} \), then \( ||x \in X : f(x) = g(x)|| < 2^{\aleph_0} \).

But 2.3 follows from a standard argument; see, for example, [8, p. 148].

2.4 Proposition. Suppose \( X \) is a Baire space of cardinality \( 2^{\aleph_0} \) such that \( |C^*(X)| = \delta X = 2^{\aleph_0} \), and either \( X \) is perfectly normal or extremally disconnected [7, exercise 1H]. Then 1.1 does not hold for \( X \).

The proof of 2.4 is similar to the proof of 2.3. If \( X \) is perfectly normal, then 2.2 is true when \( \mathcal{B} = \mathcal{T} \). And, if \( X \) is extremally disconnected, then 2.2 is true if \( \mathbb{M} = C^*(X) \) and \( f \) is assumed to be bounded on \( X \) [7, exercise 6M].

Remarks. (1) Proposition 2.4 is false if the hypothesis that \( |C^*(X)| = 2^{\aleph_0} \) is replaced by the hypothesis that \( wX = 2^{\aleph_0} \).

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(2) Statements 2.1 and 2.5 remain true if the hypothesis that \( wX = 2^{\aleph_0} \)
is replaced by the hypothesis that \( mwX = 2^{\aleph_0} \). However, if \( X \) is regular,
\( mwX = 2^{\aleph_0} \), and \( X \) satisfies the countable chain condition, then \( wX = |C^*(X)| = 2^{\aleph_0} \) (see 2.2 and 2.4 of [4]).

2.5 Proposition. Suppose \( 2^{\aleph_0} = \aleph_1 \) and that \( (X, \mathcal{F}) \) is a Baire space
that satisfies 2.1(a) and 2.1(b). If no nonempty element of \( \mathcal{F} \) is separable,
then there is a dense subspace \( Y \) of \( X \) of cardinality \( 2^{\aleph_0} \) such that
(1) \( Y \) is a Baire space that satisfies (a), (b), and (c) of 2.1, and
(2) \( Y \) is hereditarily Lindelöf.

Proof. Let \( \mathcal{B} \) be as in the proof of 2.1, and let \( \mathcal{F} \) denote the family of
all nowhere dense subsets \( F \) of \( X \) such that \( X \sim F \in \mathcal{B} \). Because \( X \) satis-
\( fies \) 2.1(a), if \( K \) is a nowhere dense subset of \( X \), then there is \( F \) in \( \mathcal{F} \) such
that \( K \subset F \) (let \( F = X \sim \gamma(X \sim cl K) \)).

Because \( |\mathcal{F}| = 2^{\aleph_0} \), we can construct, using the argument in [8, pp. 146—
147], a subset \( Y \) of \( X \) such that \( B \cap Y \neq \emptyset \) for every nonempty \( B \) in \( \mathcal{B} \) and
\( |F \cap Y| \leq \aleph_0 \) for every \( F \) in \( \mathcal{F} \). It is clear that \( Y \) is dense in \( X \) and that it satisfies \( \) (a) and \( \) (b) of 2.1. Because \( |Y \cap K| \leq \aleph_0 \) for every nowhere dense
subset \( K \) of \( X \), \( Y \) satisfies (2) and a subset of \( Y \) is nowhere dense in \( Y \) if
and only if it is countable. Hence \( Y \) is a Baire space which satisfies (c) of
2.1.

We conclude this section with some examples which illustrate the pre-
ceeding results. We assume that \( 2^{\aleph_0} = \aleph_1 \) throughout these examples.

(1) Suppose \( (X, \mathcal{F}) \) is a Souslin line. By this we mean that \( \mathcal{F} \) is the in-
terval topology induced by a total order on \( X \), and that \( (X, \mathcal{F}) \) is a compact,
connected space which satisfies the countable chain condition but which has
no nonempty separable open subsets. It is clear that \( X \) satisfies 2.1(a) and
2.1(b). By Lemma 11 of [10], \( X \) satisfies 2.1(c). Therefore Theorem 2.1 im-
pies that 1.1 does not hold for \( X \). Because \( X \) is perfectly normal, this fol-
lows from 2.4, too. \( X \) seems to be the only known example of a first count-
able, completely regular, Hausdorff, Baire space for which 1.1 does not hold.

(2) Suppose \( X \) is a quasi-regular [11, p. 164], \( T_1 \) Baire space of weight
\( 2^{\aleph_0} \) which has no isolated points, and which admits a category measure [12,
p. 156]. Then, by 2.1, any dense subset of \( X \) of cardinality \( 2^{\aleph_0} \) is a Baire
space for which 1.1 does not hold.

(2a) Let \( \mathcal{F} \) denote the density topology on the real line \( R \). It was shown
in [15] that \( (R, \mathcal{F}) \) is a Baire space for which 1.1 does not hold. This follows
from 2.1 because \( (R, \mathcal{F}) \) admits a category measure. Indeed, this follows from
2.1 even if we replace the continuum hypothesis with the hypothesis that any
subset of \( R \) of cardinality \(< 2^{\aleph_0}\) has a Lebesgue measure 0. And, if \((Y, \mathcal{U})\) is a compactification of \((R, \mathcal{J})\), then any dense subset of \( Y \) of cardinality \(2^{\aleph_0}\) is a Baire space which satisfies the hypothesis of 2.1.

(2b) Let \( S \) denote the Stone space of the Boolean algebra \( \mathcal{L}/\mathcal{J} \), where \( \mathcal{L} \) is the set of all Lebesgue measurable subsets of \([0, 1]\) and \( \mathcal{J} \) is the subset of \( \mathcal{L} \) of sets of Lebesgue measure 0. Then \( S \) is a compact, Hausdorff space which admits a category measure [11, p. 163], and which has no isolated points. Therefore 1.1 does not hold for the Baire space \( Z \) constructed in [11] such that \( Z \times Z \) is of the first category. Because \( S \) is extremally disconnected, this follows from 2.4, too.

(3) Let \( X = R \times R \) and let \( \mathcal{U} \) denote the product topology on \( X \) induced by the density topology on \( R \). Then \( X \) satisfies (a) and (b) of 2.1. However, it, does not satisfy 2.1(c) because the set \( D = \{(x, y) \in X: x - y \) is rational\} is a dense subset of \( X \) which is of the first category in \( X \). Because no nonempty open subset of \( X \) is separable, it follows from 2.5 that there is a dense subset \( Y \) of \( X \) which satisfies the hypothesis of 2.1. Note that if \( Y \) is obtained as in the proof of 2.5, then the Lebesgue measure of \( Y \) is 0. The author does not know whether 1.1 holds for \( X \).

3. Obviously, 1.1 holds for every discrete space. For pseudo-discrete spaces (\( P \)-spaces; see [7, p. 62]), the situation is more complex. As an example due to M. Henrikson shows, a \( P \)-space need not be a Baire space. And, even if the \( P \)-space \( X \) is strongly \( \alpha \)-favorable [3, p. 116], 1.1 need not hold for \( X \) (see [13]). However, if \( 2^{\aleph_0} = \mathcal{K}_1 \), then 1.1 holds for every cocompact [1, p. 292] \( P \)-space. This follows from the next statement, the proof of which is essentially the same as the proof of 1.13 of [15].

3.1 Proposition. If \( 2^{\aleph_0} = \mathcal{K}_1 \), then 1.1 holds for every quasi-regular, cocompact space \( X \) for which every nonempty \( G_\delta \) has nonempty interior.

Remark. A metrizable space is cocompact if and only if it is strongly \( \alpha \)-favorable (see Theorem 1 of [1] and Theorem 8.7 of [3]).

If \( ((X_i, \mathcal{J}_i))_{i \in I} \) is a family of topological spaces and \( m \) is an infinite cardinal number, then we denote by \( \mathcal{J}(m) \) the \( m \)-box product topology on \( X = \prod_{i \in I} X_i: i \in I \) induced by \( \mathcal{J}(m) \). The following two statements guarantee a supply of cocompact \( P \)-spaces.

3.2 Proposition. Suppose \( m \) is an infinite cardinal number such that, if \( m_n < m \) for all \( n \) in \( N \), then \( \sum \{m_n: n \in N\} < m \). If, for each \( i \) in \( I \), \( (X_i, \mathcal{J}_i) \) is a cocompact \( P \)-space, then \( (X, \mathcal{J}(m)) \) is a cocompact \( P \)-space.
3.3 Proposition. If \((X, \mathcal{J})\) is a locally compact, Hausdorff space and 
\(\mathcal{J}_\pi\) denotes the coarsest \(P\)-space topology for \(X\) containing \(\mathcal{J}\) [5, p. 55], then 
\((X, \mathcal{J}_\pi)\) is cocompact.

Proof of 3.2. It is easily verified that \((X, \mathcal{J}(m))\) is a \(P\)-space. And if 
for each \(i\) in \(I\), \(U_i\) is a compact cotopology for \(\mathcal{J}_i\), then \(\mathcal{U}(\mathcal{K}_0)\) is a compact 
cotopology for \(\mathcal{J}(m)\) [1, p. 242].

Proposition 3.3 is easily proven by using 3.11(b) of [7]. In fact, if \(\mathcal{J}\) 
is compact, then \(\mathcal{J}\) is a compact cotopology for \(\mathcal{J}_\pi\).

Proposition 3.2 provides an example of a cocompact \(P\)-space for which, 
if \(2^{\mathcal{K}_0} = 2^{\mathcal{K}_1}\), 1.1 does not hold. For let \(I\) be a set of cardinality \(\mathcal{K}_1\) and, 
for each \(i\) in \(I\), let \(\mathcal{J}_i\) denote the discrete topology on the two point set \(X_i\). 
Then Proposition 1.2 of [14] implies that 1.1 does not hold for \((X, \mathcal{J}(\mathcal{K}_1))\), 
provided \(2^{\mathcal{K}_0} = 2^{\mathcal{K}_1}\).

Remark. It is obvious that if 1.1 does not hold for a topological space 
\(X\), then it does not hold for any dense subspace of \(X\). However, it can hap- 
pen that (a) every dense subset of \(X\) of cardinality \(2^{\mathcal{K}_0}\) is a Baire space for 
which 1.1 does not hold, and (b) 1.1 holds for \(X\). For let \(X = \beta N \sim N\), where 
\(\beta N\) denotes the Stone-Cech compactification of the discrete space \(N\). By 
Proposition 1.2 of [14], (a) holds. But, if \(2^{\mathcal{K}_0} = \mathcal{K}_1\), Proposition 3.1 implies 
that (b) holds.

REFERENCES

2. J. C. Bradford and C. Goffman, Metric spaces in which Blumberg's theorem 
3. G. Choquet, Lectures on analysis. I: Integration and topological vector spaces, 
4. W. W. Comfort and Anthony W. Hager, Estimates for the number of real-valued 
5. W. W. Comfort and S. Negrepontis, Homeomorphs of three subspaces of \(\beta N\setminus N\), 
Math. Z. 107 (1968), 53–58. MR 38 #2739.
6. R. Engelking, Outline of general topology, PWN, Warsaw, 1965; English 
#5836.
7. L. Gillman and M. Jerison, Rings of continuous functions, University Ser. in 


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