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ON *-RINGS SATISFYING THE SQUARE ROOT AXIOM

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ABSTRACT. It was mentioned by Kaplansky that the parallelogram law is a key property for developing the dimension theory on the lattice of projections of a Baer *-ring, and he proved that this law follows from a pair of axioms: the EP and SR axioms. In this paper, it is shown that this law follows from only the SR axiom.

The dimension theory on the projection lattice of a Baer *-ring satisfying the EP and SR axioms were developed by Kaplansky in [3] and these arguments were extended by Berberian in [1]. For the terminology we refer to [1], and we shall generalize some theorems given in §13 of [1]. The main result of this paper is that if a Rickart *-ring A satisfies the SR axiom then the parallelogram law holds in A.

Let A be a *-ring. We denote by $P(A)$ the set of all projections of A, by $U(A)$ the set of all unitaries, and by $S(A)$ the set of symmetries (self-adjoint unitaries) $2g - 1$ with $g \in P(A)$. The parallelogram law and the square root axiom (SR axiom) are as follows (see [1, §13]):

(P) $e - e \wedge f \sim e \vee f - f$ for every $e, f \in P(A)$.

(SR) For any $x \in A$ there exists $r \in \{x^*x\}''$ such that $r^* = r$ and $x^*x = r^2$.

Now, we shall generalize Theorems 3 and 4 (Exercises 6 and 7) of [1, §13] (we remove the EP axiom and A need not be a Baer *-ring).

Theorem 1. *Let A be a Rickart *-ring. The following two statements are equivalent and they hold if A satisfies* (SR).

(α) *For every $e, f \in P(A)$ there exists $s \in S(A)$ such that $sefs = fe$.*

(β) *If $e, f \in P(A)$ are in position p' then there exists $s \in S(A)$ such that $ses = f$.*

Proof. $(\alpha) \Rightarrow (\beta)$. If $sefs = fe$ with $s \in S(A)$, then since the mapping $x \longrightarrow sxs$ is a *-automorphism of A, it is easy to verify that $s\mathrm{RP}(ef)s = \mathrm{RP}(fe)$. Since e, f are in position p', we have $\mathrm{RP}(ef) = f$ and $\mathrm{RP}(fe) = e$ by Proposition 3 of [1, §13], whence $sfs = e$ and $ses = f$.

$(\beta) \Rightarrow (\alpha)$. By proposition 5 of [1, §13], we have orthogonal decompositions $e = e' + e''$, $f = f' + f''$ with e', f' in position p' and $e''f = ef'' = 0$. It

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follows from (β) that there exists $s \in S(A)$ such that $se's = f'$, and then $se'f's = se'ssf's = f'e'$. Hence, $sefs = se'f's = f'e' = fe$.

Assume that A satisfies (SR). Putting $a = e + f - 1$, there exists $r \in \{a^2\}''$ such that $r^* = r$, $r^2 = a^2$ by (SR). Putting $b = r - a$ and $s = 1 - 2\mathrm{RP}(b)$, we have $s \in S(A)$ and we can prove that $sefs = fe$, as in the proof of Theorem (37.14) of [4] (in [4], the projection $\mathrm{RP}(b)$ is denoted by b''). Hence, (α) holds.

Theorem 2. *Let A be a Rickart *-ring. We consider the following condition*

(EU) *For every $e, f \in P(A)$ there exists $u \in U(A)$ such that $u^*efu = fe$.*

Then, we have the following implications:

$$(\mathrm{SR}) \Rightarrow (\mathrm{EU}) \Rightarrow (\mathrm{P}).$$

Proof. The first implication follows from Theorem 1. If A satisfies (EU), then for every $e, f \in P(A)$ there exists $u \in U(A)$ such that $u^*e(1-f)u = (1-f)e$. Then, $u^*\mathrm{RP}(e(1-f))u = \mathrm{RP}((1-f)e)$. By Proposition 7 of [1, §13] we have

$$\mathrm{RP}(e(1-f)) = e \vee f - f \quad \text{and} \quad \mathrm{RP}((1-f)e) = \mathrm{LP}(e(1-f)) = e - e \wedge f.$$

Hence, (P) holds. This completes the proof.

Moreover, Theorem 5 of [1, §13] also can be generalized to the case that A is a Rickart *-ring satisfying (SR).

Finally, we give some important properties of the lattice $P(A)$ which hold if A satisfies (SR).

Theorem 3. (i) *If a Rickart *-ring A satisfies* (P) *then $P(A)$ is uniform in the sense of Holland* [2].

(ii) *If a Rickart *-ring A satisfies* (EU) *then $P(A)$ is O-symmetric.*

(iii) *If a Baer *-ring A satisfies* (P) *then $P(A)$ is a dimension lattice in the sense of Remark* (35.15) *of* [4].

Proof. (i) If e and f are perspective and $ef = 0$, then since there exists $w \in A$ such that $w^*w = e$ and $ww^* = f$ by (P), putting $g = \mathrm{RP}(e + w^*) = \mathrm{RP}(f + w)$, we can show that $g \leqq e + f$, $\mathrm{UA}(e, g)$ and $\mathrm{UA}(g, f)$ by the similar way as in the proof of 4.1 of [2].

(ii) It follows from (EU) that for $e, f \in P(A)$ there exists a *-automorphism θ of A such that $\theta(ef) = fe$. Hence, $P(A)$ is O-symmetric by Theorem (37.11) and Remark (38.3) of [4] (in [4], our Rickart *-ring is called a Baer *-ring).

(iii) follows from Theorem 2.1 of [5].

REFERENCES

1. S. K. Berberian, *Baer *-rings*, Die Grundlehren der math. Wissenschaften, Band 195, Springer-Verlag, Berlin and New York, 1972.

2. S. S. Holland, Jr., *Isomorphism between interval sublattices of an orthomodular lattice*, Hiroshima Math. J. 3 (1973), 227–241.

3. I. Kaplansky, *Rings of operators*, Benjamin, New York, 1968. MR 39 #6092.

4. F. Maeda and S. Maeda, *Theory of symmetric lattices*, Die Grundlehren der math. Wissenshaften, Band 173, Springer-Verlag, Berlin, 1970. MR 44 #123.

5. S. Maeda, *On the lattice of projections of a Baer * -ring*, J. Sci. Hiroshima Univ. Ser. A 22 (1958), 75–88. MR 21 #4120.

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