

## HALL SUBGROUPS AND $p$ -SOLVABILITY

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**ABSTRACT.** Let  $G$  be a finite group and let  $\pi(G) = \{p, q_1, \dots, q_r\}$  be the set of all prime divisors of  $G$ . Suppose there is a  $p'$ -Hall subgroup  $H$ . If there are subgroups  $P, Q_1, \dots, Q_r$  such that  $P \in \text{Syl}_p(G)$ ,  $Q_i \in \text{Syl}_{q_i}(H)$ , and  $L_i = PQ_i$  is a subgroup,  $i = 1, \dots, r$ , then  $G$  is  $p$ -solvable. Moreover, if the subgroup  $H$  is solvable, then  $G$  is solvable too.

**1. Introduction.** In this paper, the following theorem will be proved.

**Theorem.** Suppose that  $G$  is a finite group and  $\pi(G) = \{p, q_1, \dots, q_r\}$  is the set of all prime divisors of  $G$ . If the following holds, then  $G$  is  $p$ -solvable.

(a) There is a  $p'$ -Hall subgroup  $H$ .

(b) There are subgroups  $P, Q_1, \dots, Q_r$  such that  $P \in \text{Syl}_p(G)$ ,  $Q_i \in \text{Syl}_{q_i}(H)$ , and  $L_i = PQ_i$  is a subgroup,  $i = 1, \dots, r$ .

Moreover, if the group  $H$  in (a) is solvable, then  $G$  is solvable too.

The group  $X = \text{PSL}(2, 7)$  has subgroups  $A, B, C$  where  $A \in \text{Syl}_2(X)$ ,  $B \in \text{Syl}_3(X)$ ,  $C \in \text{Syl}_7(X)$  such that  $AB, BC$  are subgroups of  $X$ . Of course  $X$  is not  $r$ -solvable for  $r = 2, 3$  or  $7$ .

Our notation follows Gorenstein [2]. We also let  $\text{Syl}_q(G)$  denote the set of Sylow  $q$ -subgroups of  $G$ . All groups considered in this paper are of finite order.

**2. Proof of the Theorem.** For completeness, we state the known result.

**Lemma 1 (Glauberman).** Let  $G$  be a group with  $O_p(G) \neq 1$  which is  $p$ -constrained and  $p$ -stable,  $p$  odd. If  $P \in \text{Syl}_p(G)$ , then  $G = O_p(G)N_G(ZJ(P))$ . In particular, if  $O_p(G) = 1$ , then  $ZJ(P) \triangleleft G$ .

**Proof.** See [2, p. 279].

From now on we let  $G$  be a minimal counterexample to the Theorem and use the notation  $P, H, L_i, Q_i, i = 1, \dots, r$ , which appeared in the statement of the Theorem. Since  $H$  is  $p$ -solvable,  $P$  does not normalize any nontrivial subgroup of  $H$ . Otherwise  $G$  would contain a nontrivial  $p$ -solvable normal subgroup and by induction  $G$  would not be a counterexample.

**Lemma 2.** If  $H$  is solvable, then the theorem holds.

**Proof.** Since  $H$  is solvable,  $O_{q_i}(H) \neq 1$  for some  $1 \leq i \leq r$ . Also there

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is a subgroup  $C_i$  such that  $H = Q_i C_i$ . As  $O_{q_i}(H) \subset Q_i$ , we have

$$(O_{q_i}(H))^G = (O_{q_i}(H))^{PH} = (O_{q_i}(H))^{(PQ_i)C_i} = (O_{q_i}(H))^{C_i L_i} = O_{q_i}(H)^{L_i} \subset L_i.$$

Hence  $G$  has a nontrivial normal solvable subgroup and the proof of the Theorem can be completed by induction.

**Lemma 3.**  *$P$  does not normalize any nontrivial  $p'$ -subgroup.*

**Proof.** Let  $Q$  be any  $p'$ -subgroup which is normalized by  $P$ . Set  $R = P \cdot Q$ . Then  $R = R \cap G = R \cap P \cdot H = P \cdot (R \cap H)$ . Hence  $R \cap H$  is a  $p'$ -Hall subgroup of  $R$ . Since  $Q = O_{p'}(R)$ ,  $Q = R \cap H \subseteq H$ . This implies that  $P$  normalizes the subgroup  $Q$  of  $H$ , which forces  $Q = 1$  as required.

**Lemma 4.**  $O_{q_i}(L_i) = 1, i = 1, \dots, r$ .

**Proof.** Clear from Lemma 3.

**Lemma 5.** *If  $p \neq 3$ , then the Theorem holds.*

**Proof.** If  $p = 2$ , then  $|H|$  is odd. By the Feit-Thompson theorem [1], we see that  $H$  is solvable. Lemma 2 implies the desired conclusion in this case.

Suppose  $p \geq 5$ . Then  $L_i$  is  $p$ -constrained and  $p$ -stable [2, Theorem 5.1, p. 234]. By Lemmas 1 and 4 we see that  $N_G(Z(J(P)))$  contains  $Q_i, i = 1, \dots, r$ . Hence  $Z(J(P)) \triangleleft G$  which is impossible. This completes the proof of Lemma 5.

By Lemmas 5 and 2 we may assume  $p = 3$  and  $|H|$  is even. Hence there is  $j$  such that  $q_j = 2 (1 \leq j \leq r)$ . Let  $K = \langle P, Q_i, i \neq j \rangle$ . Lemma 1 shows that  $ZJ(P) \triangleleft K$ . Hence  $N = O_3(K) \neq 1$ . Since  $|G : K|$  is a power of 2,  $G = KL_j$ . Let  $D = K \cap L_j$ . Then  $D \supseteq P \supseteq N$ . Therefore

$$N^G = N^{KL_j} = N^{L_j} \subseteq D^{L_j} \subseteq L_j.$$

Since  $L_j$  is solvable,  $N^G$  is a nontrivial normal solvable subgroup of  $G$  which is impossible. This completes the proof of the Theorem.

**Remarks.** (1) If the center of  $P$  is weakly closed in  $P$ , then it can be shown, without using the  $ZJ$  theorem, that  $G$  is  $p$ -solvable.

(2) The proof of the Theorem shows the following: Let  $S$  be a set of Sylow subgroups of  $H$  and  $p \geq 5$ . If  $H = \langle Q \mid Q \in S \rangle$  and  $L = PQ$  is a subgroup for any  $q$  in  $S$ , then  $G$  is  $p$ -solvable.

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