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HALL SUBGROUPS AND p-SOLVABILITY

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ABSTRACT. Let G be a finite group and let $\pi(G) = \{p, q_1, \ldots, q_r\}$ be the set of all prime divisors of G. Suppose there is a p'-Hall subgroup H. If there are subgroups $P, Q_1, \ldots, Q_r$ such that $P \in \mathrm{Syl}_p(G)$, $Q_i \in \mathrm{Syl}_{q_i}(H)$, and $L_i = PQ_i$ is a subgroup, $i = 1, \ldots, r$, then G is p-solvable. Moreover, if the subgroup H is solvable, then G is solvable too.

1. **Introduction.** In this paper, the following theorem will be proved.

Theorem. *Suppose that G is a finite group and $\pi(G) = \{p, q_1, \ldots, q_r\}$ is the set of all prime divisors of G. If the following holds, then G is p-solvable.*

(a) *There is a p'-Hall subgroup H.*

(b) *There are subgroups $P, Q_1, \ldots, Q_r$ such that $P \in \mathrm{Syl}_p(G)$, $Q_i \in \mathrm{Syl}_{q_i}(H)$, and $L_i = PQ_i$ is a subgroup, $i = 1, \ldots, r$.*

Moreover, if the group H in (a) *is solvable, then G is solvable too.*

The group $X = \mathrm{PSL}(2, 7)$ has subgroups A, B, C where $A \in \mathrm{Syl}_2(X)$, $B \in \mathrm{Syl}_3(X)$, $C \in \mathrm{Syl}_7(X)$ such that AB, BC are subgroups of X. Of course X is not r-solvable for $r = 2, 3$ or 7.

Our notation follows Gorenstein [2]. We also let $\mathrm{Syl}_q(G)$ denote the set of Sylow q-subgroups of G. All groups considered in this paper are of finite order.

2. **Proof of the Theorem.** For completeness, we state the known result.

Lemma 1 (Glauberman). *Let G be a group with $O_p(G) \neq 1$ which is p-constrained and p-stable, p odd. If $P \in \mathrm{Syl}_p(G)$, then $G = O_{p'}(G)N_G(ZJ(P))$. In particular, if $O_{p'}(G) = 1$, then $ZJ(P) \lhd G$.*

Proof. See [2, p. 279].

From now on we let G be a minimal counterexample to the Theorem and use the notation $P, H, L_i, Q_i, i = 1, \ldots, r$, which appeared in the statement of the Theorem. Since H is p-solvable, P does not normalize any nontrivial subgroup of H. Otherwise G would contain a nontrivial p-solvable normal subgroup and by induction G would not be a counterexample.

Lemma 2. *If H is solvable, then the theorem holds.*

Proof. Since H is solvable, $O_{q_i}(H) \neq 1$ for some $1 \leq i \leq r$. Also there

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is a subgroup C_i such that $H = Q_i C_i$. As $O_{q_i}(H) \subset Q_i$, we have

$$(O_{q_i}(H))^G = (O_{q_i}(H))^{PH} = (O_{q_i}(H))^{(PQ_i)C_i} = (O_{q_i}(H))^{C_i L_i} = O_{q_i}(H)^{L_i} \subset L_i.$$

Hence G has a nontrivial normal solvable subgroup and the proof of the Theorem can be completed by induction.

Lemma 3. *P does not normalize any nontrivial p'-subgroup.*

Proof. Let Q be any p'-subgroup which is normalized by P. Set $R = P \cdot Q$. Then $R = R \cap G = R \cap P \cdot H = P \cdot (R \cap H)$. Hence $R \cap H$ is a p'-Hall subgroup of R. Since $Q = O_{p'}(R)$, $Q = R \cap H \subseteq H$. This implies that P normalizes the subgroup Q of H, which forces $Q = 1$ as required.

Lemma 4. $O_{q_i}(L_i) = 1$, $i = 1, \ldots, r$.

Proof. Clear from Lemma 3.

Lemma 5. *If $p \neq 3$, then the Theorem holds.*

Proof. If $p = 2$, then $|H|$ is odd. By the Feit-Thompson theorem [1], we see that H is solvable. Lemma 2 implies the desired conclusion in this case.

Suppose $p \geq 5$. Then L_i is p-constrained and p-stable [2, Theorem 5.1, p. 234]. By Lemmas 1 and 4 we see that $N_G(Z(J(P))$ contains Q_i, $i = 1, \ldots, r$. Hence $Z(J(P)) \lhd G$ which is impossible. This completes the proof of Lemma 5.

By Lemmas 5 and 2 we may assume $p = 3$ and $|H|$ is even. Hence there is j such that $q_j = 2$ $(1 \leq j \leq r)$. Let $K = \langle P, Q_i i \neq j \rangle$. Lemma 1 shows that $ZJ(P) \lhd K$. Hence $N = O_3(K) \neq 1$. Since $|G:K|$ is a power of 2, $G = KL_j$. Let $D = K \cap L_j$. Then $D \supseteq P \supseteq N$. Therefore

$$N^G = N^{KL_j} = N^{L_j} \subseteq D^{L_j} \subseteq L_j.$$

Since L_j is solvable, N^G is a nontrivial normal solvable subgroup of G which is impossible. This completes the proof of the Theorem.

Remarks. (1) If the center of P is weakly closed in P, then it can be shown, without using the ZJ theorem, that G is p-solvable.

(2) The proof of the Theorem shows the following: Let S be a set of Sylow subgroups of H and $p \geq 5$. If $H = \langle Q | Q \in S \rangle$ and $L = PQ$ is a subgroup for any q in S, then G is p-solvable.

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