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EXTREMALLY DISCONNECTED SETS IN GROUPS

SADAHIRO SAEKI

ABSTRACT. It is shown that every extremally disconnected compact set in a LCA group is an SH-set.

Let G be a locally compact abelian group, and $\overline{G}$ its Bohr compactification. For a set E in G, the closure of E in $\overline{G}$ is denoted by $\overline{E}$.

We call E a set of interpolation if E has no accumulation point in G and if each bounded function on E extends to a continuous almost periodic function on G. Suppose E is such a set. Then it is obvious that $\overline{E}$ is extremally disconnected (i.e., the closure of any relatively open subset of $\overline{E}$ is relatively open in $\overline{E}$). Moreover, it is known that $\overline{E}$ is a Helson set (Kahane [1]) and also a set of uniqueness (Méla [2]).

In this note we point out the following fact.

Theorem. *Suppose that K is an extremally disconnected compact set in G. Then K is an SH-set (i.e., a set of spectral synthesis which is also a Helson set).*

Proof. If every point of K has a compact neighborhood (in K) which is an SH-set, then K is a finite union of disjoint SH-sets and is therefore an SH-set.

To force a contradiction, we assume that K contains a point x_0 such that no compact neighborhood of x_0 in K is an SH-set. We shall construct a sequence of disjoint clopen sets $A_1, B_1, A_2, B_2, \ldots$, in K as follows. First choose any disjoint clopen subsets A_1 and B_1 of K such that $x_0 \notin A_1 \cup B_1$. Suppose $A_1, B_1, \ldots, A_n, B_n$ have been chosen so that $x_0 \notin C_n = A_1 \cup B_1 \cup \cdots \cup A_n \cup B_n$. Then $K \backslash C_n$ is a clopen neighborhood of x_0 in K, which is not an SH-set. Using the characterization of SH-sets given in [3], we can therefore find two disjoint clopen subsets A_{n+1} and B_{n+1} of $K \backslash C_n$ such that $\|f\|_{A(G)} \geq n+1$ whenever $f \in A(G)$, $f = 1$ on some neighborhood of A_{n+1} in G, and $f = 0$ on some neighborhood of B_{n+1} in G. Obviously we can demand that $x_0 \notin A_{n+1} \cup B_{n+1}$. This completes the induction.

Put $A = \bigcup_1^\infty A_n$ and $B = \bigcup_1^\infty B_n \subset K$, so that A and B are disjoint open subsets of K; hence $\overline{A} \cap B = \varnothing$. Since K is extremally disconnected by hypothesis, $\overline{A}$ is open in K and therefore $\overline{A} \cap \overline{B} = \varnothing$. Consequently there

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exists a $g \in A(G)$ such that $g = 1$ on some neighborhood of $\overline{A}$ in G and $g = 0$ on some neighborhood of $\overline{B}$ in G. But then, $\|g\|_{A(G)} \geq n$ for all natural numbers n by the definitions of A and B, which is of course absurd.

This completes the proof.

Corollary. *Let E be a finite union of sets of interpolation in G. Then $\overline{E}$ is an SH-set in $\overline{G}$ and $C_0(E) \subset (M_d(\Gamma))\hat{}|_E$. Here Γ denotes the dual of G.*

Proof. This follows from our Theorem and Corollary 5.1 of [4].

Remark. In the Corollary, we cannot conclude that $l^\infty(E) = (M_d(\Gamma))\hat{}|_E$: an example appears in [5].

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DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66506

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKAZAWA, SETAGAYA, TOKYO (Current address)