EXTREMALLY DISCONNECTED SETS IN GROUPS

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ABSTRACT. It is shown that every extremally disconnected compact set in a LCA group is an SH-set.

Let $G$ be a locally compact abelian group, and $\widehat{G}$ its Bohr compactification. For a set $E$ in $G$, the closure of $E$ in $\widehat{G}$ is denoted by $\overline{E}$.

We call $E$ a set of interpolation if $E$ has no accumulation point in $G$ and if each bounded function on $E$ extends to a continuous almost periodic function on $G$. Suppose $E$ is such a set. Then it is obvious that $\overline{E}$ is extremally disconnected (i.e., the closure of any relatively open subset of $\overline{E}$ is relatively open in $\overline{E}$). Moreover, it is known that $\overline{E}$ is a Helson set (Kahane [1]) and also a set of uniqueness (Méla [2]).

In this note we point out the following fact.

**Theorem.** Suppose that $K$ is an extremally disconnected compact set in $G$. Then $K$ is an SH-set (i.e., a set of spectral synthesis which is also a Helson set).

**Proof.** If every point of $K$ has a compact neighborhood (in $K$) which is an SH-set, then $K$ is a finite union of disjoint SH-sets and is therefore an SH-set.

To force a contradiction, we assume that $K$ contains a point $x_0$ such that no compact neighborhood of $x_0$ in $K$ is an SH-set. We shall construct a sequence of disjoint clopen sets $A_1, B_1, A_2, B_2, \ldots$, in $K$ as follows.

First choose any disjoint clopen subsets $A_1$ and $B_1$ of $K$ such that $x_0 \notin A_1 \cup B_1$. Suppose $A_1, B_1, \ldots, A_n, B_n$ have been chosen so that $x_0 \notin C_n = A_1 \cup B_1 \cup \cdots \cup A_n \cup B_n$. Then $K \setminus C_n$ is a clopen neighborhood of $x_0$ in $K$, which is not an SH-set. Using the characterization of SH-sets given in [3], we can therefore find two disjoint clopen subsets $A_{n+1}$ and $B_{n+1}$ of $K \setminus C_n$ such that $\|f\|_{A(G)} \geq n + 1$ whenever $f \in A(G)$, $f = 1$ on some neighborhood of $A_{n+1}$ in $G$, and $f = 0$ on some neighborhood of $B_{n+1}$ in $G$. Obviously we can demand that $x_0 \notin A_{n+1} \cup B_{n+1}$. This completes the induction.

Put $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n \subset K$, so that $A$ and $B$ are disjoint open subsets of $K$; hence $\overline{A} \cap B = \emptyset$. Since $K$ is extremally disconnected by hypothesis, $\overline{A}$ is open in $K$ and therefore $\overline{A} \cap \overline{B} = \emptyset$. Consequently there

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exists a $g \in A(G)$ such that $g = 1$ on some neighborhood of $\overline{A}$ in $G$ and $g = 0$ on some neighborhood of $\overline{B}$ in $G$. But then, $\|g\|_{A(G)} \geq n$ for all natural numbers $n$ by the definitions of $A$ and $B$, which is of course absurd.

This completes the proof.

**Corollary.** Let $E$ be a finite union of sets of interpolation in $G$. Then $\overline{E}$ is an SH-set in $\overline{G}$ and $C_0(E) \subset (M_d(\Gamma))^\sim_E$. Here $\Gamma$ denotes the dual of $G$.

**Proof.** This follows from our Theorem and Corollary 5.1 of [4].

**Remark.** In the Corollary, we cannot conclude that $l^\infty(E) = (M_d(\Gamma))^\sim_E$: an example appears in [5].

**REFERENCES**